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Characterization of Variance Optimal Equivalent Local Martingale Measure and Stochastic Volatility Model with Small Diffusion Coefficient

T. Toronjadze[1,2]

[1]Georgian American University, Business School, 10 Merab Aleksidze Str., 0160, Tbilisi, Georgia;
[2]A. Razmadze Mathematical Institute of I. Javakhishvili Tbilisi State University, 2 Merab Alekside II Lane, 0193 Tbilisi, Georgia

Abstract

Characterization of variance optimal equivalent local martingale measure plays key role in several important problems of statistics of random processes. For stochastic volatility model with small diffusion coefficient the given characterization is used for robust statistic purposes.

Key words and phrases: Stochastic volatility, small diffusion coefficient, variance optimal equivalent local martingale measure.

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1 A financial market model

Let $(\Omega, \mathcal{F}, F = (\mathcal{F}_t)_{0\leq t\leq T}, P)$ be a filtered probability space with filtration F satisfying the usual conditions, where $T \in (0, \infty]$ is a fixed time horizon. Assume that $\mathcal{F}_0$ is trivial and $\mathcal{F}_T = \mathcal{F}$.

There exist $d+1$, $d \geq 1$, primitive assets: one bond, whose price process is assumed to be 1 at all times and d risky assets (stocks), whose R^d-valued price process $X = (X_t)_{0\leq t\leq T}$ is a continuous semimartingale given by the relation:

$$dX_t = \mathrm{diag}(X_t)\, dR_t, \quad X_0 > 0, \tag{1.1}$$

where $\mathrm{diag}(X)$ denotes the diagonal $d \times d$-matrix with diagonal elements $X^1, \ldots, X^d$, and the yield process $R = (R_t)_{0\leq t\leq T}$ is a R^d-valued continuous semimartingale satisfying the structure condition (SC). That is (see Schweizer [1]),

$$dR_t = d\langle \widetilde{M}\rangle_t \lambda_t + d\widetilde{M}_t, \quad R_0 = 0, \tag{1.2}$$

where $\widetilde{M} = (\widetilde{M})_{0\leq t\leq T}$ is a R^d-valued continuous martingale, $\widetilde{M} \in \mathcal{M}^2_{0,loc}(P)$, $\lambda = (\lambda_t)_{0\leq t\leq T}$ is a F-predictable R^d-valued process, and the mean-variance tradeoff (MVT) process $\widetilde{\mathcal{K}} = (\widetilde{\mathcal{K}})_{0\leq t\leq T}$ of process R

$$\widetilde{\mathcal{K}}_t := \int_0^t \lambda'_s d\langle \widetilde{M}\rangle_s \lambda_s = \langle \lambda' \cdot \widetilde{M}\rangle_t < \infty, \quad P\text{-a.s.}, \;\; t \in [0,T]. \tag{1.3}$$

Remark. Remember that all vectors are assumed to be column vectors.

Suppose that the martingale $\widetilde{M}$ has the form

$$\widetilde{M} = \sigma \cdot M, \tag{1.4}$$

where $M = (M_t)_{0\leq t\leq T}$ is a R^d-valued continuous martingale, $M \in \mathcal{M}^2_{0,loc}(P)$, $\sigma = (\sigma_t)_{0\leq t\leq T}$ is a $d\times d$-matrix valued, F-predictable process with $\mathrm{rank}(\sigma_t) = d$ for any t, P-a.s., the process $(\sigma_t^{-1})_{0\leq t\leq T}$ is locally bounded, and

$$\langle \widetilde{M}\rangle_T = \int_0^T \sigma_t \langle M\rangle_t \sigma'_t < \infty. \quad P\text{-a.s.} \tag{1.5}$$

Assume now that the following condition is satisfied: there exist fixed R^d-valued, F-predictable process $k = (k_t)_{0\leq t\leq T}$ such that

$$\lambda = \lambda(\sigma) = (\sigma')^{-1} k. \tag{1.6}$$

In this case, from (1.2) we get

$$\begin{aligned} dR_t = d\langle \widetilde{M}\rangle_t \lambda_t + d\widetilde{M}_t &= \sigma_t d\langle M\rangle_t \sigma'_t (\sigma'_t)^{-1} k_t + \sigma_t dM_t \\ &= \sigma_t (d\langle M\rangle_t k_t + dM_t) \end{aligned} \tag{1.7}$$

and

$$\widetilde{\mathcal{K}}_t = \int_0^t \lambda_s' d\langle\widetilde{M}\rangle_s \lambda_s = \int_0^t k_t'((\sigma_t')^{-1})'\sigma_t d\langle M\rangle_t \sigma_t'(\sigma_t')^{-1} k_t$$
$$= \int_0^t k_t' d\langle M\rangle_t k_t = \langle k\cdot M\rangle_t := \mathcal{K}_t.$$

From (1.3) we have

$$\mathcal{K}_t < \infty, \quad P\text{-a.s. for all } t\in[0,T]. \tag{1.8}$$

Thus, if we introduce the process $M^0 = (M_t^0)_{0\le t\le T}$ by the relation

$$dM_t^0 = d\langle M\rangle_t k_t + dM_t, \quad M_0^0 = 0, \tag{1.9}$$

then the MVT process $\mathcal{K} = (\mathcal{K}_t)_{0\le t\le T}$ of R^d-valued semimartingale M^0 is finite, and hence M^0 satisfies SC.

Finally, the scheme (1.1), (1.2), (1.4), (1.6) and (1.9) can be rewritten in the following form

$$\begin{gathered} dX_t = \operatorname{diag}(X_t)\, dR_t, \quad X_0 > 0, \\ dR_t = \sigma_t dM_t^0, \quad R_0 = 0, \\ dM_t^0 = d\langle M\rangle_t k_t + dM_t, \quad M_0 = 0, \end{gathered} \tag{1.10}$$

where σ and k satisfy (1.5) and (1.8), respectively.

This is our financial market model.

2 Characterization of variance-optimal equivalent local martingale measure

A key role in mean-variance hedging plays variance-optimal equivalent local martingale measure (ELMM) (see, e.g., [2, 3, 4]).

We start with remark that the sets of ELMMs for processes X, R and M^0 of form (1.10) coincide. Hence we can and will consider the simplest process M^0.

Introduce the notation

$$\mathcal{M}_2^e := \left\{ Q\sim P \;:\; \frac{dQ}{dP}\in L^2(P),\;\; M^0 \text{ is a } Q\text{-local martingale}\right\},$$

and suppose that

$$(c.1) \qquad \mathcal{M}_2^e \neq \emptyset.$$

The solution $\widetilde{P}$ of the optimization problem

$$E\mathcal{E}_T^2(\mathcal{M}^Q) \to \inf_{Q\in\mathcal{M}_2^e}$$

is called variance-optimal ELMM.

Here

$$\frac{dQ}{dP}\Big|_{\mathcal{F}_T} = \mathcal{E}_T(M^Q),$$

and $(\mathcal{E}_t(M^Q))_{0\le t\le T}$ is the Dolean exponential of martingale M^Q.

It is well-known (see, e.g, Schweizer [1, 5]) that under condition (c.1) variance-optimal ELMM $\widetilde{P}$ exist.

Denote

$$\widetilde{z}_T := \frac{d\widetilde{P}}{dP}\Big|_{\mathcal{F}_T},$$

and introduce RCLL process $\widetilde{z} = (\widetilde{z}_t)_{0\le t\le T}$ by the relation

$$\widetilde{z}_t = E^{\widetilde{P}}(\widetilde{z}_T/\mathcal{F}_t), \quad 0 \le t \le T.$$

Then, by Schweizer [1, 5],

$$\widetilde{z}_t = \widetilde{z}_0 + \int_0^T \zeta_t'\, dM_t^0, \tag{2.1}$$

where $\zeta = (\zeta_t)_{0\le t\le T}$ is the R^d-valued F-predictable process with

$$\int_0^T \zeta_t'\, d\langle M\rangle_t \zeta_t < \infty,$$

and the process $(\int_0^t \zeta_s' dM_s^0)_{0\le t\le T}$ is a $\widetilde{P}$-martingale.

Relation (2.1) easily implies that the process $\widetilde{z}$ is actually continuous.

Suppose, in addition to (c.1), that the following conditions is satisfied:

$(c.*)$ All P-local martingales are continuous.

This technical assumption is satisfied in stochastic volatility models, where $F = F^w$ is the natural filtration generated by the Wiener process.

It shown in Mania and Tevzadze [6], Mania et al. [7] that, under conditions (c.1) and (c.*), density $\widetilde{z}_T$ of variance optimal ELMM is uniquely characterized by the relation

$$\widetilde{z}_T = \frac{\mathcal{E}_T((\varphi - k)' \cdot M^0)}{E\mathcal{E}_T((\varphi - k)' \cdot M^0)}\,, \tag{2.2}$$

where φ together with the pair (L, c) is the unique solution of the following equation

$$\frac{\mathcal{E}_T((\varphi - 2k)' \cdot M)}{\mathcal{E}_T(L)} = c\mathcal{E}_T^2(-k' \cdot M), \tag{2.3}$$

where $L \in M^2_{0,loc}(P)$, $\langle L, M\rangle = 0$, c is a constant.

Moreover, the process $\zeta = (\zeta_t)_{0\le t\le T}$ from (2.1) has the form

$$\zeta_t = (\varphi_t - k_t)\mathcal{E}_t((\varphi - k)' \cdot M^0). \tag{2.4}$$

Here $\varphi = (\varphi_t)_{0\le t\le T}$ is a R^d-valued, F-predictable process with

$$\int_0^T \varphi_t'\, d\langle M\rangle_t \varphi_t < \infty.$$

Let τ be F-stopping time.

Denote $\langle k' \cdot M\rangle_{T\tau} = \langle k' \cdot M\rangle_T - \langle k' \cdot M\rangle_\tau$.

Theorem (see also Biagini et al. [8], Laurent and Pham [9]).

1. *Equation* (2.3) *is equivalent to equation*

$$\frac{\mathcal{E}_T(\varphi' \cdot M^*)}{\mathcal{E}_T(L)} = ce^{\langle k\cdot M\rangle_T}, \tag{2.5}$$

where the R^d-valued process $M^* = (M^*_t)_{0\le t\le T}$ is given by the relation

$$dM^*_t = 2d\langle M\rangle_t k_t + dM_t, \quad M^*_0 = 0.$$

2. a) *If there exists the martingale $m = (m_t)_{0\le t\le T}$, $m \in \mathcal{M}^2_{0,loc}(P)$ such that*

$$e^{-\langle k'\cdot M\rangle_T} = c + m_T, \quad \langle m, M\rangle = 0, \tag{2.6}$$

then $\varphi \equiv 0$ and $L_T = \int_0^t \frac{1}{c+m_s} dm_s$ solve the equation (2.5).

In this case

$$\widetilde{z}_T = \frac{\mathcal{E}_T(-k' \cdot M^0)}{E\mathcal{E}_T(-k' \cdot M^0)}\,, \tag{2.7}$$

process $\zeta = (\zeta_t)_{0\le t\le T}$ *from* (2.1) *is equal to*

$$\zeta_t = -k_t \mathcal{E}_t(-k' \cdot M^0),$$

and

$$E\Big[\Big(\frac{\widetilde{z}_T}{\widetilde{z}_\tau}\Big)^2 \Big/ \mathcal{F}_\tau\Big] = \frac{1}{E(e^{-\langle k'\cdot M\rangle_{T\tau}}/\mathcal{F}_\tau)}\,.$$

b) *If there exist* R^d*-valued* F*-predictable process* $\ell = (\ell_t)_{0\le t\le T}$, $\int_0^T \ell_t' d\langle M\rangle_t \ell_t < \infty$, *and*

$$e^{\langle k'\cdot M\rangle_T} = c + \int_0^T \ell_t'\, dM_t^*,$$

then $L \equiv 0$ *and* $\varphi_t = \frac{\ell_t}{c+\int_0^t \ell_s' dM_s^*}$ *solve the equation* (2.5).

In this case,

$$\widetilde{z}_T = \mathcal{E}_T(-k'\cdot M) \ \ (:= \widehat{z}_T, \ \ \textit{the density of minimal martingale measure } \widetilde{P}),$$

and

$$E\bigg(\Big(\frac{\widetilde{z}_T}{\widetilde{z}_\tau}\Big)^2 \Big/ \mathcal{F}_\tau\bigg) = E^{P^*}\big(e^{\langle k'\cdot M\rangle_{T\tau}}/\mathcal{F}_\tau\big),$$

where $dP^* = \mathcal{E}_T(-2k'\cdot M)dP$.

Proof. 1. By the Yor formula

$$\mathcal{E}_T\big((\varphi-2k)'\cdot M\big) = \mathcal{E}_T(\psi'\cdot M - 2k'\cdot M)$$
$$= \mathcal{E}_T\bigg(\varphi'\cdot\bigg(M + 2\int_0^\cdot d\langle M\rangle_t k_t\bigg) - 2\int_0^\cdot \psi_t' d\langle M\rangle_t k_t - 2k'\cdot M\bigg)$$
$$= \mathcal{E}_T(\varphi'\cdot M^*)\mathcal{E}_T(-2k'\cdot M),$$

and

$$\mathcal{E}_T^2(-k'\cdot M) = \mathcal{E}_T(-2k'\cdot M)e^{\langle k'\cdot M\rangle_T},$$

Assertion follows.

2. a) Note at first that $\langle L, M\rangle = 0$. Further, by the Îto formula we can write

$$\ln(c+m_t) - \ln c = \int_0^t \frac{1}{c+m_s}\,dm_s - \frac{1}{2}\int_0^t \frac{1}{(c+m_s)^2}\,dm_s\,.$$

Hence

$$e^{\ln(c+m_T)-\ln c} = \mathcal{E}_T(L)$$

and thus

$$\mathcal{E}_T(L) = \frac{c+m_T}{c} = \frac{e^{-\langle k'\cdot M\rangle_T}}{c}\,.$$

Finally, by the Bayes rule and the Girsanov Theorem,

$$E\biggl(\Bigl(\frac{\widetilde{z}_T}{\widetilde{z}_\tau}\Bigr)^2 \Big/ \mathcal{F}_\tau\biggr) = \frac{E(\mathcal{E}_T(-2k'\cdot M)e^{-\langle k'\cdot M\rangle_T}/\mathcal{F}_\tau)}{E^2(\mathcal{E}_T(-k'\cdot M)e^{-\langle k'\cdot M\rangle_T}/\mathcal{F}_\tau)}$$

$$= \frac{E^*(c+m_T/\mathcal{F}_\tau)}{(\widehat{E}(c+m_\tau/\mathcal{F}_\tau))^2}\,\frac{\mathcal{E}_T^2(-k'\cdot M)}{\mathcal{E}_T(-2k'\cdot M)} = \frac{c+m_\tau}{(c+m_\tau)^2}\cdot e^{\langle k'\cdot M\rangle_\tau}$$

$$= \frac{1}{E(e^{-\langle k'\cdot M\rangle_{T\tau}}/\mathcal{F}_\tau)}\,.$$

The proof of case 2 b) is quite analogous. □

3 Stochastic volatility model with small diffusion coefficient

Denote by $\mathrm{Ball}_L(0,r)$, $r\in[0,\infty)$, the closed r-radius ball in the space $L = L_\infty(dt\times dP)$, with the center at the origin, and let

$$\mathcal{H} := \bigl\{h=\{h_{ij}\},\ \ i,j,=\widehat{1,d}\ :\ h\ \text{ is } F\text{-predictable, } d\times d\text{-matrix valued process},\ \ \mathrm{rank}(h)=d,\ \ h_{ij}\in \mathrm{Ball}_L(0,r),\ \ r\in[0,\infty)\bigr\}.$$

The class $\mathcal{H}$ is called the class of alternatives.

Fix the value of small parameter $\delta>0$, as well as $d\times d$-matrix valued F-predictable process $\sigma^0=(\sigma^0_t)_{0\le t\le T}$, $\mathrm{rank}(\sigma^0)=d$, with

$$\int_0^T \sigma^0_t d\langle M\rangle_t(\sigma^0_t)' < \infty\ \ P\text{-a.s.}$$

Denote

$$A_\delta = \{\sigma\ :\ \sigma=\sigma^0+\delta h,\ \ h\in\mathcal{H}\}.$$

Consider the set of processes $\{R^\sigma$ (or X^σ), $\sigma\in A_\delta\}$, which represents the misspecification of asset price model.

For simplicity, consider the one-dimensional case ($d = 1$).

Let $a(t, y)$ be a drift coefficient of volatility process. Consider the stochastic volatility model with misspecified asset price model and fully specified volatility process model with small diffusion coefficient ε,

$$dX_t = X_t\, dR_t, \quad X_0 > 0,$$
$$dR_t = (\sigma_t^0 + \delta h_t) dM_t^0, \quad R_0 = 0,$$
$$dY_t = a(t, Y_t) dt + \varepsilon dw_t^\sigma, \quad Y_0 = 0, \quad 0 \le t \le T,$$

where

$$dM_t^0 = k_t dt + dw_t^R,$$

$h \in \mathcal{H}$ and σ_t^0 is the center of the confidence interval of volatility, which shrinks to

$$\sigma_t = f^{\frac{1}{2}}(Y_t).$$

Here, $f(\cdot)$ is a continuous one-to-one positive locally bounded function (e.g., $f(x) = e^x$).

Note that to robust statistics problems for stochastic volatility model with small randomness are already devoted our papers [10, 11].

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