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Optimal harvest problem based on the size structured stochastic model

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We consider the problem of finding the optimal harvesting effort that gives the maximal yield for the size-structured population model. A stochastic growth equation is given whose solution density satisfies size-structured population growth equation and boundary conditions.

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1 Introduction

During the last years, structured population models have played a significant role in the mathematical analysis and control of populations in biology and demography (see [3]). Among the individual structure, there are many structural differences, such as age, body size, gender, gene, and life stage. Our aim is to study an optimal harvest problem for size-structured population model. There is no standard optimization technique for such models and their investigation is highly nontrivial. Here we use dynamic programming technique for the solution for some kind of harvest problems.

The remaining parts of this paper are organized as follows. In Section 2, we give an exactly solvable examples of optimal harvesting. In Section 3 the general case for non-structured population model is considered. In Section 4 we formulate a stochastic control problem equivalent to optimal harvest problem for size-structured population model.

2 Examples of optimal harvesting

Let us consider optimization problem for the fish population

$$\dot{N}(t) = (\lambda(t) - \alpha(t))N(t),$$
$$\max_{\alpha} \int_0^T p(t)\alpha(t)N(t)dt,\ 0 \le \alpha(t) \le 1,$$

where $\lambda(t)-$ is fecundity mines mortality, $\alpha(t)$ is catchability, $p(t)$ is mean price of fish. Let

$$V(n,t) = \max_{\alpha} \int_t^T p(s)\alpha(s)N(n,s)ds,$$
$$N(n,s) = n + \int_t^T (\lambda(s) - \alpha(s))N(n,s)ds.$$

The Bellman equation for this problem is

$$V_t(n,t) + \max_{\alpha}[(\lambda(t) - \alpha)nV_n(n,t) + p(t)\alpha n] = 0,\ V(T,n) = 0.$$

For $V(n,t) = \varphi(t)n$ we get

$$\varphi'(t) + \lambda(t)\varphi(t) + (p(t) - \varphi(t))^+ = 0,\ \varphi(T) = 0.$$

The solution is

$$\varphi(t) = \begin{cases} \int_t^T e^{\int_t^s (\lambda(r)-1)dr} p(s)ds,\ t > t^* \\ e^{\int_t^{t^*} \lambda(r)dr},\ t < t^*, \end{cases}$$

where t^* is the root of equation $\int_{t^*}^T e^{\int_{t^*}^s (\lambda(r)-1)dr} p(s)ds = p(t^*)$. When $\lambda \equiv m, p(t) \equiv 1$ this equation becomes $\frac{1}{m-1}(e^{(T-t^*)(m-1)} - 1) = 1$ or $T - t^* = \frac{\ln(m)}{m-1}$. The optimal control is

$$\alpha^*(t) = I(p(t) - V_n > 0) = I(p(t) > \varphi(t)) = I(t^* < t \le T).$$

For the stochastic problem

$$dN(t) = (\lambda(t) - \alpha(t))N(t)dt + \sigma(t)N(t)dW(t),$$
$$\max_{\alpha} E\int_0^T p(t)\alpha(t)N(t)dt,\ 0 \le \alpha(t) \le 1$$

we get $\alpha^*(n,t) = I_{(p(t)>V_n(n,t))}$

$$V_t(n,t) + \max_{\alpha}[(\lambda(t)-\alpha)nV_n(n,t) + p(t)\alpha n] + \frac{1}{2}\sigma^2 n^2 V_{nn}(n,t) = 0,\ V(T,n) = 0$$

or

$$V_t(n,t) + n[\lambda(t)V_n(n,t) + (p(t) - V_n(n,t))^+] + \frac{1}{2}\sigma^2 n^2 V_{nn}(n,t) = 0,\ V(T,n) = 0.$$

Again solution is of the form $V(n,t) = n\varphi(t)$, where

$$\varphi'(t) + \lambda(t)\varphi(t) + (p(t) - \varphi(t))^+ = 0,\ \varphi(T) = 0.$$

Hence $\alpha^*(n,t) = \alpha^*(t) = I_{(p(t)>\varphi(t))}$.

3 The general case

Let us consider the optimal harvest problem

$$dN(t) = (\lambda(N,t) - \alpha(N,t))N(t)dt + \sigma(N,t)N(t)dW(t),$$
$$\max_{0\le\alpha(t)\le c} E\int_0^T p(N,t)\alpha(t)N(t)dt,$$

where price process has the form $dp(N,t) = \alpha^p(N,t)dt + \sigma^p(N,t)dW_t$.
Since

$$\begin{aligned} E\int_0^T p(N,t)\alpha(N,t)N(t)dt \\ = E\int_0^T p(N,t)\lambda(N,t)N(t)dt - E\int_0^T p(N,t)dN(t) \\ = E\int_0^T p(N,t)\lambda(N,t)N(t)dt - E(P(N,T)N(T) \\ + \int_0^T N(t)\alpha^p(N,t)dt + \int_0^T \sigma(N,t)\sigma^p(N,t)N(t)dt) \\ = E\eta(N), \end{aligned}$$

where

$$\eta(N) = \int_0^T p(N,t)\lambda(N,t)N(t)dt - p(N,T)N(T)$$
$$+ \int_0^T N(t)\alpha^p(N,t)dt + \int_0^T \sigma(N,t)\sigma^p(N,t)N(t)dt$$

we get

$$dN(t) = \sigma(N,t)N(t)((\lambda(N,t) - \alpha(N,t))\sigma(N,t)^{-1}dt + dW(t))$$
$$= \sigma(N,t)N(t)d\tilde{W}(t),$$
$$\max_\alpha E\eta(N),\ 0 \le \alpha(t) \le c.$$

Let N^0 be solution of $dN_t^0 = \sigma(N^0,t)N_t^0 dW_t$. Then by Girsanov Theorem

$$E\eta(N) = E\eta(N^0)\mathcal{E}_T\left(\int_0^{\cdot} (\lambda(N^0,s) - \alpha(N^0,s))\sigma^{-1}(N^0,s)dW_s\right) =: E^\alpha\eta(N^0).$$

The value process $V_t = \operatorname{ess\,sup}_\alpha E^\alpha(\eta|F_t)$ satisfies

$$dV(t) = -\max_{0\le a\le c}((\lambda(N^0,t) - a)\sigma^{-1}(N^0,t)Z_t)dt + Z_t dW(t),$$
$$V_T = \eta(N^0),$$

or

$$dV(t) = -\lambda(N^0,t)\sigma^{-1}(N^0,t)Z_t dt - c\sigma^{-1}(N^0,t)Z_t^- dt + Z_t dW(t),$$
$$V_T = \eta(N^0),\ \ \alpha^*(t) = I_{(Z_t>0)}.$$

Denoting $\tilde{Z}_t = (N_t^0)^{-1}\sigma^{-1}(N^0,t)Z_t$ one gets

$$dV(t) = -N_t^0(\lambda(N^0,t)\tilde{Z}_t + c\tilde{Z}_t^-)dt + \tilde{Z}_t dN^0(t),$$
$$V_T = \eta(N^0).$$

Since $N^0\sigma^{-1}$ is positive we get $\alpha^*(t) = I_{(\tilde{Z}_t>0)}$.

4 A size structured population model and optimal harvest problem

Let $N(t) = \int_0^\infty u(x,t)dx$ be the decomposition of population by the size. Then in [1],[2] the optimal harvest problem for the system

$$\partial_t u(x,t) + \partial_x(g(x)u(x,t)) + m(x)u(x,t) = -\mu(t)u(x,t), \tag{1}$$

$$u(x,0) = u_0(x), \tag{2}$$

$$g(0)u(0,t) = \int_0^\infty \beta(t,x)u(x,t)dx. \tag{3}$$

under revenue function

$$\int_0^T \int_0^\infty \mu(t)u(x,t)dxdt = \int_0^T \mu(t)N(t)dt \overset{\mu}{\to} \max \tag{4}$$

were studied. Here $m(x), \beta(t,x)$ denote natural mortality and fecundity of population of size x, $\mu(t)$ denotes fishing mortality and $g(x)$ is rate of growth of individuals.

Our aim is to show that the solution of (1)-(3) may be represented by the distribution function of a stochastic processes.

Let $\eta_k, k = 1,2,\dots$ be i.i.d. nonnegative random variables with probability density $f(x), x \geq 0$, $\sigma_n = \sum_k^n \eta_k$ and η_0 a independent random variable with probability density $f_0(x)$. Denote $N(t) = \#\{n : \eta_0 + \sigma_n \leq t\}$. Suppose $A(t) = t - \sigma_{N_t}$ and $a(t,x) = P(A(t) \leq x)$.

A) Let $u(x) = \sum_{n=1}^\infty f^{n*}(x)$ and f_0 are differentiable and $\tilde{u} = f_0 + u * f$, where $*$ denotes convolution.

Lemma 1. Let condition A) is satisfied. Then equation $E\Phi(A(t)) = \int_0^\infty \Phi(x)(1-F(x))\tilde{u}(t-x)dx$ is satisfied for every continuous bounded function Φ.

Proof. For each bounded, continuous function Φ we get

$$E\Phi(A_t) = \sum_{n=0}^\infty E\Phi(t-\tau_n)I_{(\tau_n \leq t < \tau_{n+1})}$$

$$= \sum_{n=0}^\infty E\Phi(t-\tau_n)I_{(t_n \leq t < \tau_{n+1})}(1-F(t-\tau_n))$$

$$= \sum_{n=0}^\infty \int_0^t \Phi(t-s)(1-F(t-s))dF^{*n}(s).$$

$$= \int_0^t \Phi(t-s)(1-F(t-s))dU(s) = \int_0^t \Phi(s)(1-F(s))u(t-s)ds.$$

$\square$

Let $x(t)$ be the solution of growth equation

$$\dot{x}(t) = g(x(t)), x(0) = 0.$$

Let $Y(t) = \begin{cases} x(t - \tau_{N_t}), & t < \zeta \\ \partial, & t \geq \zeta \end{cases}$, where $\zeta = \inf\{s; \int_0^s \mu(v)dv > \tau\}$ and τ independent, exponentially distributed r. v. with parameter 1. Then Y satisfy the equation

$$Y(t) = Y(0) + \int_0^t g(Y(s))ds - \int_0^t Y(s-)dN(s),\ t < \zeta.$$

For each bounded, continuous function φ one obtains

$$\begin{aligned}\int_0^\infty \varphi(r)\rho(t,r)dr = E\varphi(Y(t)), t < \zeta) = E\varphi(A(t)), t < \zeta) \\ = E\varphi(x(t - \tau_{N_t}))e^{-\int_0^t \mu(v)dv}.\end{aligned}$$

Using lemma 1 for the function $\Phi(t) = \varphi(x(t))$ we get

$$\begin{aligned}E\varphi(x(t - \tau_{N_t}))e^{-\int_0^t \mu(v)dv} &= e^{-\int_0^t \mu(v)dv} \int_0^t \Phi(s)(1 - F(s))\tilde{u}(t - s)ds \\ &= e^{-\int_0^t \mu(v)dv} \int_0^\infty \varphi(x(s))(1 - F(s))\tilde{u}(t - s)/g(x(s))dx(s) \\ &= e^{-\int_0^t \mu(v)dv} \int_0^\infty \varphi(r)(1 - F(x^{-1}(r)))\tilde{u}(t - x^{-1}(r))/g(r)dr\end{aligned}$$

and $\rho(t,r)e^{\int_0^t \mu(v)dv} = (1 - F(x^{-1}(r)))\tilde{u}(t - x^{-1}(r))/g(r)$. Hence

$$\begin{aligned}\partial_t\rho(t,r)e^{\int_0^t \mu(v)dv} + \mu(t)\rho(t,r)e^{\int_0^t \mu(v)dv} \\ +\partial_r(g(r)\rho(t,r))e^{\int_0^t \mu(v)dv} \\ = \partial_t(\rho(t,r)e^{\int_0^t \mu(v)dv}) + \partial_r(g(r)\rho(t,r)e^{\int_0^t \mu(v)dv}) \\ = (1 - F(x^{-1}(r)))\tilde{u}'(t - x^{-1}(r))/g(r) \\ -(1 - F(x^{-1}(r)))\tilde{u}'(t - x^{-1}(r))x^{-1\prime}(r) \\ -f(x^{-1}(r)))\tilde{u}(t - x^{-1}(r))x^{-1\prime}(r) \\ = -f(x^{-1}(r)))\tilde{u}(t - x^{-1}(r))/g(r) \\ = -\frac{f(x^{-1}(r)))}{1 - F(x^{-1}(r))}e^{\int_0^t \mu(v)dv}\rho(t,r).\end{aligned}$$

Integrating equation $\partial_t\rho(t,r)+\mu(t)\rho(t,r)+\partial_r(g(r)\rho(t,r))+\frac{f(x^{-1}(r)))}{1-F(x^{-1}(r))}\rho(t,r) = 0$ and using $\int_0^\infty \rho(t,r)dr = e^{-\int_0^t \mu(v)dv}$ one obtains

$$g(0)\rho(t,0) = \int_0^\infty \frac{f(x^{-1}(r))}{1-F(x^{-1}(r))}\rho(t,r)dr.$$

Finally we get

Proposition 1. Let condition A) is satisfied. Then

$$\partial_t\rho(t,r) + \partial_r(g(r)\rho(t,r)) + \mu(t)\rho(t,r) + \frac{f(x^{-1}(r)))}{1-F(x^{-1}(r))}\rho(t,r) = 0,$$
$$g(0)\rho(t,0) = \int_0^\infty \beta(r)\rho(t,r)dr,$$

where $\beta(r) = \frac{1}{1-F(x^{-1}(r))}(f(x^{-1}(r))$.

Corollary. The function $u(x,t) = N_0\rho(t,x)$ satisfy equations (3).

The optimal harvesting problem (1)-(4) can be rewritten as the problem

$$Y(t) = Y(0) + \int_0^t g(Y(s))ds - \int_0^t Y(s-)dN(s),\ t < \zeta(\alpha),$$
$$\max_\alpha E\int_0^{T\wedge\zeta(\alpha)} \pi(t,Y(t))\alpha(t,Y(t))dt.$$

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