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Black-Scholes Model and Martingale functions of a Brownian Motion

B. Chikvinidze[1)], M. Mania[2)] and R. Tevzadze[3)]

[1)] Georgian-American University and Institute of Cybernetics of Georgian Technical University, Tbilisi, Georgia
(e-mail: beso.chiqvinidze@gmail.com)
[2)] Razmadze Mathematical Institute of Tbilisi State University and Georgian-American University, Tbilisi, Georgia,
(e-mail: misha.mania@gmail.com)
[3)] Georgian-American University and Institute of Cybernetics of Georgian Technical University, Tbilisi, Georgia
(e-mail: rtevzadze@gmail.com)

Abstract. Using a description of time-dependent martingale functions of a Brownian Motion and some natural assumptions (axioms), we show that the evolution of the asset price process should follow to the geometric Brownian Motion.

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Let $W = (W_t, t \geq 0)$ be a standard Brownian Motion defined on a complete probability space $(\Omega, \mathcal{F}, \mathcal{P})$. Denote by $F = (\mathcal{F}_t, t \geq 0)$ an augmented filtration generated by the Brownian Motion W.

Let consider the Black-Scholes model of financial market, which consists of two securities. The first security, called a bond, with the price $(B_t, t \geq 0)$

defined by the differential equation

$$dB_t = rB_t dt, \quad B_0 = b, \quad r > 0, b > 0 \tag{1}$$

and the second security, a stock, whose price process satisfies the stochastic differential equation

$$dS_t = \mu S_t dt + \sigma S_t dW_t, \quad S_0 = z > 0, \tag{2}$$

where μ and σ are constants $\mu, \sigma \in R, \sigma \neq 0$.

The solution of (1) is a continuously compound interest

$$B_t = be^{rt}, \quad r > 0,$$

which is the amount to which a capital x increases during the time interval t by interest compounding.

The unique solution of SDE (2) is represented as

$$S_t = ze^{\sigma W_t + (\mu - \frac{\sigma^2}{2})t}, \quad S_0 = z. \tag{3}$$

Let $F(t, z)$ be the amount to which a capital z increases during the time interval t by continuously compounding.

It was shown in J. Aczel [1] by using Cauchy's functional equations and some natural assumptions (axioms), that $F(t, x)$ should be equal to be^{rt}.

Assume that

1) the capital $F(z, t)$ does not change by dividing the original capital $z = x + y$ into separate investments x, y. This assumption yields

$$F(t, x + y) = F(t, x) + F(t, y), \quad t, x, y \in (0, \infty)$$

On the other hand, it is also natural to assume

2) that the capital that is evaluated during the time t from the initial capital x to $F(t, x)$ will amount to just as much after passing of the additional time s, as if the initial sum of money had been evaluated for the time $s + t$, i.e.,

$$F(t + s, x) = F(s, F(t, x)), \quad s, t, x \in (0, \infty).$$

It follows from assumption 1) and from the Cauchy additive functional equation that

$$F(t, x) = xC(t)$$

for some positive function $C(t), t \geq 0$.

Assumption 2) implies that

$$xC(t+s) = F(t,x)C(s) = xC(t)C(s),$$

and hence $C(t)$ satisfies an exponential functional equation

$$C(t+s) = C(t)C(s)$$

the general measurable positive solution of which is expressed as $C(t) = \exp rt$, for some $r \in R$. Thus,

$$F(t,x) = xe^{rt}, \quad r \in R$$

and the following theorem is valid

Theorem (J. Aczel 1966). The general positive solution of the system of functional equations

$$F(t,x+y) = F(t,x)+F(t,y), \quad F(t+s,x) = F(s,F(t,x)), \quad s,t,x,y \in (0,\infty),$$

which is increasing in t is represented in the form

$$F(t,x) = xe^{rt}$$

for some $r > 0$.

We shall use similar arguments (as in J. Aczel [1]) to derive the representation (3) of the stock price process.

Let $f(t,z,w)$ be a measurable function of three variables and let $f(t,z,W_t)$ describes the price of a stock at time $t \geq 0$ with initial price z.

If we suppose that one can sell or buy any part of the stock, it will be natural to assume that $f(t,z,W_t)$ to which the initial price (capital) z evaluates during the time interval t, does not change by dividing the original initial capital $z = x+y$ into separate investments x, y. This yields

Assumption 1). for any $t,x,y \in (0,\infty)$

$$f(t,x+y,W_t) = f(t,x,W_t) + f(t,y,W_t).$$

It is also natural to assume that the capital that is evaluated during the time s from the initial capital x to $f(s,x,W_s)$ will amount to just as much after passing of the additional time t, as if the initial sum of money had been evaluated for the time $s+t$; thus by independent increment property of the Brownian Motion W and that $\tilde{W}_t = W_{t+s} - W_s, t \geq 0$ is a Brownian Motion starting from zero at the moment $t = 0$, we shall have that

Assumption 2). for any $s, t, x \in (0, \infty)$

$$f(s+t, z, W_{s+t}) = f(t, f(s, z, W_s), W_{t+s} - W_s). \tag{4}$$

The proof of the main theorem is based on the following proposition proved in [2].

Proposition 1. Let $(C(t,x), t \geq 0, x \in R)$ be a continuous strictly positive function such that $C(0,0) = 1$. Then the processes

$$N_t(1) = \frac{C(t, W_t)}{EC(t, W_t)}, \quad and \quad N_t(2) = \frac{C^2(t, W_t)}{EC^2(t, W_t)}, \quad t \geq 0,$$

are strictly positive martingales if and only if the function C is of the form

$$C(t, x) = e^{cx+bt}, \quad \text{for some constants } \ b, c \in R. \tag{5}$$

We apply this assertion to a justification of the model (2).

Theorem 1. Let $f(t, z, w)$ be a measurable function of three variables which is continuous at t and let $S_t = f(t, z, W_t), S_0 = z$ describes the price of a stock at time $t \geq 0$ with initial price z. Assume that

$$A1) \qquad f(t, x+y, W_t) = f(t, x, W_t) + f(t, y, W_t) \quad \text{for any} \quad t, x, y \in (0, \infty)$$

$$A2) \quad f(s+t, x, W_{s+t}) = f(t, f(s, x, W_s), W_{t+s} - W_s). \quad \text{for any} \quad t, x, y \in (0, \infty).$$

Then

$$S_t = S_0 e^{cW_t + bt},$$

which is the same as (3) with $c = \sigma$ and $b = \mu - \sigma^2/2$.

Proof. According to the general measurable solution of the Cauchy additive functional equation it follows from Assumption 1) that

$$f(t, z, W_t) = C(t, W_t)z \tag{6}$$

and by Assumption 2)

$$zC(s+t, W_{s+t}) = f(s, z, W_s)C(t, W_{s+t} - W_s)$$

and, hence

$$C(s+t, W_{s+t}) = C(s, W_s)C(t, W_{s+t} - W_s). \tag{7}$$

Changing the time parameter we rewrite this equation in more convenient for us form

$$C(t, W_t) = C(s, W_s)C(t-s, W_t - W_s). \tag{8}$$

Since $W_t - W_s$ is independent of W_s, taking expectations in (8) we have that

$$EC(t, W_t) = EC(t-s, W_t - W_s)EC(s, W_s). \tag{9}$$

If we take the conditional expectations in the same equality (8), having in mind that $W_t - W_s$ is independent of $\mathcal{F}_s$, we obtain

$$E(C(t, W_t)|\mathcal{F}_s) = f(s, W_s)EC(t-s, W_t - W_s) \quad P - a.s.. \tag{10}$$

Therefore, substituting the expression of $EC(t-s, W_t - W_s)$ from (9) into (10) we get the martingale equality

$$E\Big(\frac{C(t, W_t)}{EC(t, W_t)}\Big|\mathcal{F}_s\Big) = \frac{C(s, W_s)}{EC(s, W_s)}, \quad P - \text{a.s..}$$

The martingale property of the process $N_t(2)$ is proved similarly, if we use equality

$$C^2(s, W_s)C^2(t-s, W_t - W_s) = C^2(t, W_t) \quad P - a.s. \tag{11}$$

instead of equality (8).

Therefore, it follows from Propositions 1 that the function $C(t, x)$ is of the form

$$C(t, x) = e^{cx+bt}$$

and

$$S_t = S_0 e^{cW_t + bt},$$

which is the same as (3) with $c = \sigma$ and $b = \mu - \sigma^2/2$.

Finally we notice that using the stochastic flow approach and under stronger regularity assumptions (see, H. Kunita 1990 [3]) it is also possible to derive equation (2) from the semi-group property (4).

References

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