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A stochastic model of predator-prey population dynamics

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Abstract

Abstract. We present results of an analysis for a randomized three-dimensional predator-prey model representing the dynamics of wolf-deer interactions.

1 Introduction

We study a predator – prey stochastic model (initiated by R. Chitashvili) in discrete and continuous time. For this we define the transition probabilities and Markov chain realization by the random difference schemes as well as the systems of stochastic differential equations. We show that in case of scaling, the solution of the system approaches the solution of the equation of ordinary differential equation. We present the graphs of these solutions for the specific parameter set and different initial conditions. Graphs illustrate the equilibrium points which the system approaches in infinite time.

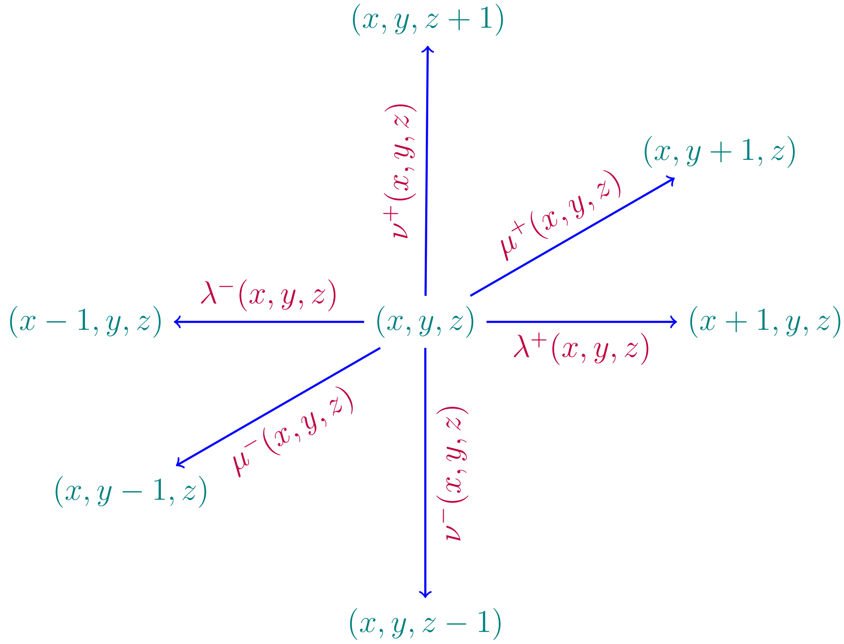
2 The discrete time Model

Let us consider two populations: deer and wolves and assume that deer divide into the group of strong and week ones. Denote by x, y, z the number of strong deer, week deer and wolves respectively.

Let $\lambda^{\pm}(x,y,z), \mu^{\pm}(x,y,z), \nu^{\pm}(x,y,z),\ (x,y,z) \in N^3$ be intensity of transitions

$$(x,y,z) \to (x \pm 1, y, z), (x,y,z) \to (x, y \pm 1, z), (x,y,z) \to (x, y, z \pm 1)$$

and $\lambda = \lambda^+ + \lambda^-, \mu = \mu^+ + \mu^-, \nu = \nu^+ + \nu^-$ respectively .



Let us introduce the following notations

$$p^{\pm}(x,y,z) = \frac{\lambda^{\pm}(x,y,z)}{(\lambda+\mu+\nu)(x,y,z)},$$
$$q^{\pm}(x,y,z) = \frac{\mu^{\pm}(x,y,z)}{(\lambda+\mu+\nu)(x,y,z)},$$
$$r^{\pm}(x,y,z) = \frac{\nu^{\pm}(x,y,z)}{(\lambda+\mu+\nu)(x,y,z)},$$
$$p = p^+ + p^-, q = q^+ + q^-, r = r^+ + r^-.$$

Then recurrent equations

$$\begin{aligned} X_{n+1} &= X_n + I_{(\varepsilon_n < p^+(X_n,Y_n,Z_n))} - I_{(p^+(X_n,Y_n,Z_n)) \le \varepsilon_n < p(X_n,Y_n,Z_n)}, \qquad (1) \\ Y_{n+1} &= Y_n + I_{(p(X_n,Y_n,Z_n) \le \varepsilon_n < (p+q^+)(X_n,Y_n,Z_n))} - I_{(p+q^+)(X_n,Y_n,Z_n)) \le \varepsilon_n < (p+q)(X_n,Y_n,Z_n))}, \\ Z_{n+1} &= Z_n + I_{((p+q)(X_n,Y_n,Z_n) \le \varepsilon_n < (p+q+r^+)(X_n,Y_n,Z_n))} - I_{((p+q+r^+)(X_n,Y_n,Z_n) \le \varepsilon_n < 1)}, \end{aligned}$$

where ε_n is i.i.d. with uniform distribution, define Markov chain with such transition probabilities. Indeed

$$P(X_{n+1} = X_n \pm 1, Y_{n+1} = Y_n, Z_{n+1} = Z_n | X_n, Y_n, Z_n) = P(X_{n+1} = X_n \pm 1 | X_n, Y_n, Z_n)$$
$$= P(I_{(\varepsilon_n < p^+(X_n,Y_n,Z_n))} - I_{(p^+(X_n,Y_n,Z_n)) \leq \varepsilon_n < p^+(X_n,Y_n,Z_n) + p^-(X_n,Y_n,Z_n))} = \pm 1) | X_n, Y_n, Z_n)$$
$$= P(\varepsilon_n < p^{\pm}(X_n, Y_n, Z_n) | X_n, Y_n, Z_n) = p^{\pm}(X_n, Y_n, Z_n)$$

and at cetera.

Assume that the share of strong deer increases proportionally to fraction of strong deer pairs into all deer pairs, i.e. by intensity $(x + y)\frac{x^2}{(x+y)^2}$. Similarly for week deers we get intensity $(x+y)(1-\frac{x^2}{(x+y)^2})$. The mortality of strong and week deer is defined as $d^s x + e^s xz,\ d^w y + e^w yz$ respectively. We define the rate of fecundity and mortality of wolfs as: $\alpha z + \beta'(x + y),\ \delta z + \gamma' \frac{z^2}{x+y}$. Therefore

$$\lambda^+(x, y, z) = \frac{x^2}{x + y},\ \lambda^-(x, y, z) = (d^s x + e^s xz),$$
$$\mu^+(x, y, z) = \frac{(x + y)^2 - x^2}{x + y},\ \mu^-(x, y, z) = (d^w y + e^w yz),$$
$$\nu^+(x, y, z) = \alpha z + \beta'(x + y),\ \nu^-(x, y, z) = \delta z + \gamma' \frac{z^2}{x + y}.$$

We take $e^s = d^s = 0.2,\ e^w = d^w = 0.5,\ \alpha = \delta,\ \beta' = 1, \gamma' = 1$ and $(X_0, Y_0, Z_0) = (8, 8, 8)$. We graphically present the solution of (1) below

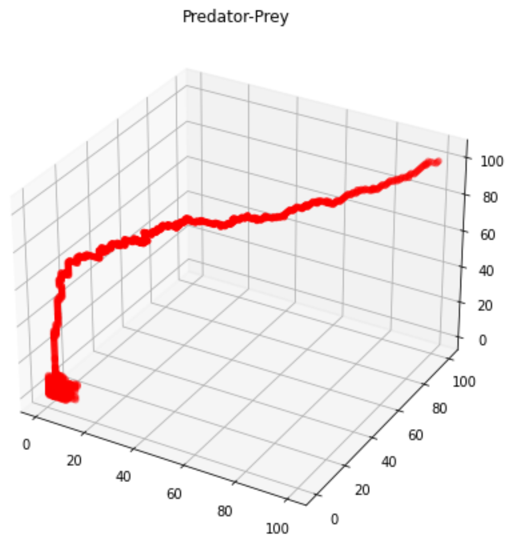


Figure 1: graph of random dynamics

3 The continuous time Model

Let $\mathcal{N}(duds)$ be the Poisson point process driven by Lebesgue measure $duds$ and $\tilde{\mathcal{N}}(duds) = \mathcal{N}(duds) - duds$. Then the Markov chain in continuous time is defined by the SDE

$$
\begin{aligned}
X_t &= X_0 + \int_0^t \int_0^\infty (I_{(u<\lambda^+(X_{s-},Y_{s-},Z_{s-}))} - I_{(\lambda^+(X_{s-},Y_{s-},Z_{s-}))u<\lambda(X_{s-},Y_{s-},Z_{s-})})\mathcal{N}(duds), \\
Y_t &= Y_0 + \int_0^t \int_0^\infty (I_{(\lambda(X_{s-},Y_{s-},Z_{s-})\leq u<(\lambda+\mu^+)(X_{s-},Y_{s-},Z_{s-}))} \\
&\quad -I_{(\lambda+\mu^+)(X_{s-},Y_{s-},Z_{s-}))\leq u<(\lambda+\mu)(X_{s-},Y_{s-},Z_{s-}))})\mathcal{N}(duds), \\
Z_t &= Z_0 + \int_0^t \int_0^\infty (I_{((\lambda+\mu)(X_{s-},Y_{s-},Z_{s-})\leq u<(\lambda+\mu+\nu^+)(X_{s-},Y_{s-},Z_{s-}))} \\
&\quad -I_{((\lambda+\mu+\nu^+)(X_{s-},Y_{s-},Z_{s-})\leq u<(\lambda+\mu+\nu)(X_{s-},Y_{s-},Z_{s-}))})\mathcal{N}(duds).
\end{aligned}
$$

Then

$$\begin{aligned}
X_t &= X_0 + \int_0^t (\lambda^+(X_{s-}, Y_{s-}, Z_{s-})) - \lambda^-(X_{s-}, Y_{s-}, Z_{s-}))ds + L(t),\\
Y_t &= Y_0 + \int_0^t (\mu^+(X_{s-}, Y_{s-}, Z_{s-}) - \mu^+(X_{s-}, Y_{s-}, Z_{s-}))ds + M(t)\\
Z_t &= Z_0 + \int_0^t (\nu^+(X_{s-}, Y_{s-}, Z_{s-}) - \nu^+(X_{s-}, Y_{s-}, Z_{s-}))ds + N(t),
\end{aligned}$$

where

$$\begin{aligned}
L(t) &= \int_0^t \int_0^\infty (I_{(u<\lambda^+(X_{s-},Y_{s-},Z_{s-}))} - I_{(\lambda^+(X_{s-},Y_{s-},Z_{s-}))u<\lambda(X_{s-},Y_{s-},Z_{s-}))})\tilde{\mathcal{N}}(duds)\\
M(t) &= \int_0^t \int_0^\infty (I_{(\lambda(X_{s-},Y_{s-},Z_{s-})\leq u<(\lambda+\mu^+)(X_{s-},Y_{s-},Z_{s-}))}\\
&\quad -I_{(\lambda+\mu^+)(X_{s-},Y_{s-},Z_{s-}))\leq u<(\lambda+\mu)(X_{s-},Y_{s-},Z_{s-}))})\tilde{\mathcal{N}}(duds)\\
N(t) &= \int_0^t \int_0^\infty (I_{((\lambda+\mu)(X_{s-},Y_{s-},Z_{s-})\leq u<(\lambda+\mu+\nu^+)(X_{s-},Y_{s-},Z_{s-}))}\\
&- I_{((\lambda+\mu+\nu^+)(X_{s-},Y_{s-},Z_{s-})\leq u<(\lambda+\mu+\nu)(X_{s-},Y_{s-},Z_{s-}))})\tilde{\mathcal{N}}(duds).
\end{aligned}$$

Obviously $\langle M, L\rangle = \langle M, N\rangle = \langle L, N\rangle = 0$ and

$$\begin{aligned}
\langle L\rangle_t &= \int_0^t (\lambda^+(X_{s-}, Y_{s-}, Z_{s-})) + \lambda^-(X_{s-}, Y_{s-}, Z_{s-}))ds = \int_0^t \lambda(X_{s-}, Y_{s-}, Z_{s-})ds,\\
\langle M\rangle_t &= \int_0^t (\mu^+(X_{s-}, Y_{s-}, Z_{s-})) + \mu^-(X_{s-}, Y_{s-}, Z_{s-}))ds = \int_0^t \mu(X_{s-}, Y_{s-}, Z_{s-})ds,\\
\langle N\rangle_t &= \int_0^t (\nu^+(X_{s-}, Y_{s-}, Z_{s-})) + \nu^-(X_{s-}, Y_{s-}, Z_{s-}))ds = \int_0^t \nu(X_{s-}, Y_{s-}, Z_{s-})ds.
\end{aligned}$$

Let $\lambda_K^\pm(i), \mu_K^\pm(i), \nu_K^\pm(i)$ denote $K\lambda^\pm(i/K), K\mu^\pm(i/K), K\nu_K^\pm(i/K), i = 0, 1, \ldots,$ and let (X^K, Y^K, Z^K) be the corresponding solution. Let $(\tilde{X}^K, \tilde{Y}^K, \tilde{Z}^K) = (X^K, Y^K, Z^K)/K$. The following proposition follows from results of [2] .

Proposition 1. Let $\lambda_K^\pm(i), \mu_K^\pm(i), \nu_K^\pm(i)$ be Lipschitzz continuous and nonnegative on $\mathbb{R}_+^3$. Then $(\tilde{X}^K, \tilde{Y}^K, \tilde{Z}^K) \overset{K\to\infty}{\to} (x, y, z)$ in law in $D[0, T]^3$, where (x, y, z) is solution the of ODE

$$\begin{aligned}
\dot{x} &= \lambda^+(x, y, z) - \lambda^-(x, y, z),\\
\dot{y} &= \mu^+(x, y, z) - \mu^-(x, y, z),\\
\dot{z} &= \nu^+(x, y, z) - \nu^-(x, y, z).
\end{aligned}$$

In our model

$$\lambda^+ = \frac{X^2}{X+Y},\ \lambda^- = (d^s X + e^s XZ),$$
$$\mu^+ = \frac{(X+Y)^2 - X^2}{X+Y},\ \mu^- = (d^w Y + e^w YZ),$$
$$\nu^+ = \alpha Z + \beta'(X+Y),\ \nu^- = \delta Z + \gamma' \frac{Z^2}{X+Y}.$$

Therefore

$$dX_t = \left(\frac{X^2}{X+Y} - (d^s X + e^s XZ)\right) dt + dL(t)$$
$$dY_t = \left(\frac{(X+Y)^2 - X^2}{X+Y} - (d^w Y + e^w YZ)\right) dt + dM(t)$$
$$dZ_t = \left(\alpha Z + \beta'(X+Y) - \delta Z + \gamma' \frac{Z^2}{X+Y}\right) dt + dN(t).$$

In deterministic case one obtains

$$\begin{aligned}
\dot{x} &= \frac{x^2}{x+y} - (d^s x + e^s xz), \\
\dot{y} &= \frac{(x+y)^2 - x^2}{x+y} - (d^w y + e^w yz), \\
\dot{z} &= \alpha z + \beta'(x+y) - \delta z - \gamma' \frac{z^2}{x+y}.
\end{aligned} \tag{2}$$

Such type of population model was studied in [1].

Remark. If $\tilde{y} = x + y$ then

$$\begin{aligned}
\dot{x} &= \frac{x^2}{\tilde{y}} - x(d^s + e^s z), \\
\dot{\tilde{y}} &= \tilde{y}(1 - d^w - e^w z) + x(d^w + e^w z - d^s - e^s z), \\
\dot{z} &= (\alpha - \delta) z + \beta' \tilde{y} - \gamma' \frac{z^2}{\tilde{y}}.
\end{aligned}$$

In particular case of parameters we have

$$
\begin{aligned}
\dot{x} &= \frac{x^2}{x+y} - 0.2x(1+z), \\
\dot{y} &= \frac{(x+y)^2 - x^2}{x+y} - 0.5y(1+z), \\
\dot{z} &= (x+y) - \frac{z^2}{x+y}.
\end{aligned} \tag{3}
$$

or for $(x, \tilde{y}, z) = (x, x+y, z)$

$$
\begin{aligned}
\dot{x} &= \frac{x^2}{\tilde{y}} - 0.2x(1+z), \\
\dot{\tilde{y}} &= \tilde{y} - (0.5\tilde{y} - 0.3x)(1+z), \\
\dot{z} &= \tilde{y} - \frac{z^2}{\tilde{y}}.
\end{aligned}
$$

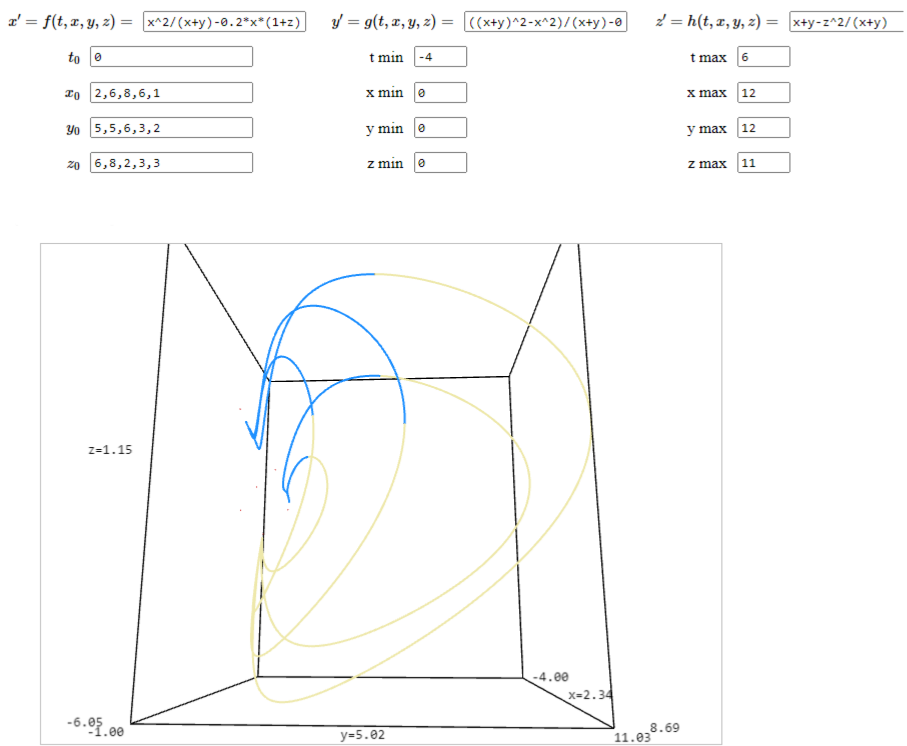


Figure 2: phase portrait of system (3)

The solution of

$$
\begin{aligned}
0 &= \frac{x^2}{x+y} - 0.2x(1+z), \\
0 &= \frac{(x+y)^2 - x^2}{x+y} - 0.5y(1+z), \\
0 &= (x+y) - \frac{z^2}{x+y}.
\end{aligned}
$$

defines the equilibrium points of the population system, which are marked in Figure 2 by red points.

References

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