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A Generalization of the von Bertalanffy growth Model using the BSDE Approach

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Abstract. The generalized von Bertalanffy growth model with random extremal length is expressed as a unique solution of a Backward Stochastic Differential Equation.

Keywords: Bertalanffy growth model, Levy Process, Brownian Motion, Backward Equation.

1 Introduction

We shall use stochastic exponentials and Backward Stochastic Differential Equation (BSDE) approach to generalize the fish growth deterministic model of von Bertalanffy [4], which is most commonly used as a descriptive model of size-at age data.

The von Bertalanffy model, which is written for the case of decreasing growth with age is a differential equation with a linearly decreasing growth rate, or

$$\frac{dL}{dt} = K(L_\infty - L_t), \quad L_\infty > 0 \quad K > 0, \tag{1}$$

with initial condition $L(t_0) = L_0$. The solution to differential equation (1) is

$$L_t = L_\infty[1 - e^{-K(t-t_0)}] + L_0 e^{-K(t-t_0)}, \tag{2}$$

where

L_∞ - is the upper bound for the variable under study, that can only be reached after infinity time,

K - is the curvature parameter, or von Bertalanffy growth rate, that determines the speed with which the fish attains L_∞.

t_0 - determines the time at which the fish has a size equal to zero and could be negative.

For simplicity we assume that $L_0 = 0$ and $t_0 = 0$ which means that an individual would have been of length 0 at age 0. This results in the "LVB" growth model

$$L_t = L_\infty[1 - e^{-Kt}]. \tag{3}$$

Several stochastic growth models are available in the literature. Some individual-based stochastic models of growth (see, e.g., [1], [2]) are proposed using stochastic differential equations of the type

$$L_t = L_0 + \int_0^t a(s, L_s)ds + \int_0^t \sigma(s, L_s)dW_s,$$

where L_t is the size at time t, $a(t, L_t)$ characterizes the deterministic intrinsic growth of the individual, $\sigma(t, L_t)$ gives the magnitude of the random fluctuations and W_t is a Brownian Motion.

In Russo et al [3] the growth model of fish (and other animals) as a solution of linear stochastic differential equation driven by a Levy process with positive jumps (a subordinator) was proposed, the unique solution of which is the stochastic exponential of the Levy process. This model admits a certain number of desirable features and it is the first stochastic model with increasing paths, giving more realistic stochastic model of individual growth.

The model proposed in Russo et al [3] is given by the process Y_t, which is obtained as the solution of the stochastic differential equation (SDE)

$$dY_t = (L_\infty - Y_{t-})dX_t \tag{4}$$

with initial condition $L_0 = 0$, where X_t is a subordinator. Note that a subordinator is a Levy process with increasing paths.

If the process X cannot make jumps larger than 1 (which is natural to assume in this context), then solution of this equation is

$$L_t = L_\infty(1 - \mathcal{E}_t(-X)), \tag{5}$$

where $\mathcal{E}_t(-X)$ is the stochastic exponent of the process $-X$ and the extreme length L_∞ is assumed to be a constant. Note that L_t defined by this model is increasing process and it coincides with the von Bertalanffy growth curve when X is a deterministic subordinator $X_t = kt$.

This approach (as all existing) has a drawback as a growth model, since the asymptotic length of the fish is assumed to be a constant. This implies that the variation of fish length tends to zero, which is not realistic, as it would imply that all individuals should reach the same limiting size. In order to overcome this problem it seems natural to assume that the extremal size of a fish is itself a random variable, thus accounting for the individual variability. Therefore, it is natural to use Backward SDE's (instead of the forward SDEs) with the random boundary condition at the end equal to the asymptotic length of a fish.

To generalize the von Bertalanffy model when the extreme length L_∞ is a random variable, let first consider the simple case and only assume that L_∞ is a bounded random variable measurable with respect to $F^W_\infty = \vee_{t\geq 0} F^W_t$, where W is a Brownian Motion and $(F^W_t, t \geq 0)$ is the filtration generated by W.

We write this model as a solution of the Backward Stochastic Differential equation (BSDE)

$$Y_t = \int_0^t Y_s \frac{Ke^{-Ks}}{1 - e^{-Ks}} ds + \int_0^t Z_s dW_s, \tag{6}$$

with the boundary condition

$$Y_\infty = \lim_{t\to\infty} Y_t = L_\infty. \tag{7}$$

The solution process to equation (6)-(7) is

$$L_t = E(L_\infty | F^W_t)[1 - e^{-Kt}]. \tag{8}$$

More exactly the solution of (6)-(7) is a pair (Y_t, Z_t)

$$Y_t = L_t, \qquad Z_t = \varphi_t(1 - e^{-Kt}),$$

where L_t is defined by (8) and φ_t is the integrand from the integral representation of the martingale

$$E(L_\infty|F_t^W) = EL_\infty + \int_0^t \varphi_s dW_s,$$

which can be immediately verified by the integration by part formula.

Note that, since (8) implies

$$EL_t = EL_\infty[1 - e^{-Kt}],$$

the expectation of L_t follows the Von Bertalanffy-type pattern with L_∞ replaced by EL_∞.

Remark that, if in (8) instead of exponential distribution function $1-e^{-Kt}$ we shall take general continuous distribution function $G(t)$, then the process $L_t = E(L_\infty|F_t^W)G(t)$ will satisfy the BSDE

$$Y_t = \int_0^t \frac{Y_s}{G(s)} dG(s) + \int_0^t Z_s dW_s, \tag{9}$$

with the same boundary condition (7).

We shall generalize expression (5) (see Theorem 1) assuming that L_∞ is a random variable and consider this variable as a boundary condition at infinity of a BSDE for L_t driven by a subordinator X and a Brownian Motion W, independent of X. The linear BSDEs derived in the paper differ from classical cases by considering not integrable coefficients on the infinite time interval. Under additional assumption that the extreme size L_∞ of a fish is a random variable measurable with respect to the σ-algebra F_∞^W generated by the Brownian Motion W, i.e., when two sources of randomness, the random individual variability (related with L_∞) and the environmental randomness (related with the process X_t), are independent, the BSDE takes simpler and more natural form (see Corollary 1).

2 The main results.

Let $X = (X_t, t \geq 0)$ be a Levy process with affine process $\alpha t, \alpha > 0$, with zero Brownian part and with positive jumps (a subordinator). Let

$W = (W_t, t \geq 0)$ be a Brownian Motion. Suppose that X and W are independent processes defined on a complete probability space $(\Omega, \mathcal{F}, P)$ with filtration $F = (F_t, t \geq 0)$ generated by W and X and let

$$F_\infty = \vee_{t\geq 0} F_t.$$

Suppose that in the model (3) or (5) L_∞ is an integrable F_∞-measurable random variable. Then L_t should be a random process and if we assume that the process L_t is adapted with respect to the filtration F_t, from (5) we obtain that

$$L_t = E(L_\infty|F_t)[1 - \mathcal{E}_t(-X)]. \tag{10}$$

Let consider the growth model given by expression (10) and let's see what equation this process satisfies.

Let μ be the measure of jumps of the process X and let $\tilde{\mu}$ be its compensator. Note that in our case the compensator is of the form

$$\tilde{\mu}(ds, dx) = \nu(dx)ds,$$

where $\nu(dx)$ is the Levy measure on $R_+ =]0, \infty[$ with $\int_{R_+}(1 \wedge x^2)\nu(dx) < \infty$.

We recall that the Levy process is a cadlag process with stationary independent increaments, hence all jumps of X are totally inaccesible.

Denote by H the expression

$$H(s,x) = \frac{x\mathcal{E}_{s-}(-X)}{1 + (x-1)\mathcal{E}_{s-}(-X)}.$$

Let consider the following linear BSDE (backward stochastic differential equation)

$$Y_t = \int_0^t \int_{R_+} (Y_s + K(s,x))H(s,x)\nu(dx)ds + \alpha \int_0^t Y_s \frac{\mathcal{E}_{s-}(-X)}{1 - \mathcal{E}_{s-}(-X)} ds +$$

$$+ \int_0^t \int_{R_+} K(s,x)(\mu - \tilde{\mu})(dx, ds) + \int_0^t Z_s dW_s, \tag{11}$$

with the boundary condition

$$Y_\infty = \lim_{t\to\infty} Y_t = L_\infty. \tag{12}$$

Definition. Let $\mathcal{V}$ be the class of cadlag processes $(Y_t, t \geq 0)$, such that the family of random variables $(Y_\tau, \tau \in \mathcal{T})$ is uniformly integrable, where $\mathcal{T}$ is the set of stopping times.

Theorem 1. Let X be a Levy process with increasing paths (a subordinator) and let $\Delta X_t < 1$ for all $t \geq 0$. Assume that L_∞ is an integrable F_∞-measurable random variable.

Then the process

$$L_t = E(L_\infty|F_t)[1 - \mathcal{E}_t(-X)] \quad t \geq 0 \tag{13}$$

is the unique solution of the BSDE (11)-(12) in the class $\mathcal{V}$.

Proof. The boundary condition follows from the Levy theorem, since $\lim_{t\to\infty} \mathcal{E}_t(-X) = 0$ and

$$\lim_{t\to\infty} L_t = \lim_{t\to\infty} E(L_\infty|F_t) \lim_{t\to\infty}(1 - \mathcal{E}_t(-X)) = L_\infty.$$

It follows from (13) that the process L_t is a special semimartingale and let

$$L_t = A_t + M_t, \qquad A_0 = 0, M_0 = 0 \tag{14}$$

be the canonical decomposition, where A is the predictable process of finite variation and M is a local martingale, which by the integral representation property can be expressed as

$$M_t = \int_0^t \int_{R_+} K(s,x)(\mu - \tilde{\mu})dxds + \int_0^t Z_s dW_s \tag{15}$$

for some predictable Z and K with

$$\int_0^t Z_s^2 ds < \infty, \quad \int_0^t \int_{R_+} K^2(s,x)\nu(dx)ds < \infty \quad a.s.$$

First note that, since $\Delta\mathcal{E}_t(-X) = -\mathcal{E}_{t-}(-X)\Delta X_t$, we have for any $0 < r < t$

$$\frac{1}{1 - \mathcal{E}_t(-X)} - \frac{1}{1 - \mathcal{E}_r(-X)} =$$

$$= \sum_{r<s\leq t} \left(\frac{1}{1 - \mathcal{E}_{s-}(-X)} - \frac{1}{1 - \mathcal{E}_{s-}(-X)} \right) - \int_r^t \frac{\alpha \mathcal{E}_s(-X)}{(1 - \mathcal{E}_s(-X))^2} ds =$$

$$= -\sum_{r<s\le t} \frac{\mathcal{E}_{s-}(-X)\Delta X_s}{\big(1-\mathcal{E}_{s-}(-X)\big)\big(1-\mathcal{E}_{s-}(-X)+\mathcal{E}_{s-}(-X)\Delta X_s\big)} - \int_r^t \frac{\alpha \mathcal{E}_s(-X)}{(1-\mathcal{E}_s(-X))^2} ds =$$

$$= -\int_r^t \int_{R_+} \frac{\mathcal{E}_{s-}(-X)x}{\big(1-\mathcal{E}_{s-}(-X)\big)\big(1-\mathcal{E}_{s-}(-X)+\mathcal{E}_{s-}(-X)x\big)} \mu(dxds) - \int_r^t \frac{\alpha \mathcal{E}_s(-X)}{(1-\mathcal{E}_s(-X))^2} ds$$

$$= -\int_r^t \int_{R_+} \frac{H(s,x)}{1-\mathcal{E}_{s-}(-X)} \mu(ds,dx) - \int_r^t \frac{\alpha \mathcal{E}_s(-X)}{(1-\mathcal{E}_s(-X))^2} ds. \tag{16}$$

By the Itô formula and (16) for any $r>0$

$$\frac{L_t}{1-\mathcal{E}_t(-X)} - \frac{L_r}{1-\mathcal{E}_r(-X)} = \tag{17}$$

$$= \int_r^t \frac{1}{1-\mathcal{E}_{s-}(-X)} dA_s + \int_r^t \frac{1}{1-\mathcal{E}_{s-}(-X)} dM_s +$$

$$+ \int_r^t L_{s-} d(1-\mathcal{E}_{s-}(-X))^{-1}) + [L,(1-\mathcal{E}(-X))^{-1}]_t - [L,(1-\mathcal{E}(-X))^{-1}]_r =$$

$$= \int_r^t \frac{1}{1-\mathcal{E}_{s-}(-X)} dA_s + \int_r^t \frac{1}{1-\mathcal{E}_{s-}(-X)} dM_s - \int_r^t \frac{\alpha L_s \mathcal{E}_{s-}(-X)}{(1-\mathcal{E}_{s-}(-X))^2} ds$$

$$- \int_r^t \int_{R_+} L_{s-} \frac{H(s,x)}{1-\mathcal{E}_{s-}(-X)} \mu(ds,dx) - \int_r^t \int_{R_+} K(s,x) \frac{H(s,x)}{1-\mathcal{E}_{s-}(-X)} \mu(ds,dx)$$

$$- \int_r^t \int_{R_+} \Delta A_s \frac{H(s,x)}{1-\mathcal{E}_{s-}(-X)} \mu(ds,dx).$$

Since A is predictable and all jumps of X are totally inaccessible

$$\int_r^t \int_{R_+} \Delta A_s \frac{H(s,x)}{1-\mathcal{E}_{s-}(-X)} \mu(ds,dx) = 0$$

and if we isolate in (17) the martingale part, we obtain that

$$\frac{L_t}{1-\mathcal{E}_t(-X)} - \frac{L_r}{1-\mathcal{E}_r(-X)} = \tag{18}$$

$$= \int_r^t \frac{1}{1-\mathcal{E}_{s-}(-X)} dA_s - \int_r^t \frac{\alpha L_s \mathcal{E}_{s-}(-X)}{(1-\mathcal{E}_{s-}(-X))^2} ds =$$

$$- \int_r^t \int_{R_+} (L_s + K(s,x)) \frac{H(s,x)}{1-\mathcal{E}_{s-}(-X)} \nu(dx)ds \ + \ \text{local martingale.}$$

Since by (13) the left-hand side of (18) is a martingale on the interval $[r,\infty]$ for any $r>0$, the bounded variation part in (18) should be equal to zero. Therefore,

$$A_t - A_r = \alpha \int_r^t L_s \frac{\mathcal{E}_s(-X)}{1-\mathcal{E}_{s-}(-X)} ds + \tag{19}$$

$$+ \int_r^t \int_{R_+} (L_s + K(s,x))H(s,x)\nu(dx)ds$$

for any $r>0$. Since $A_0=0$ the process A is a cadlag process of finite variation, passing to the limit as r tends to 0 we obtain from (19) that

$$A_t = \int_0^t \int_{R_+} (L_s + K(s,x))H(s,x)\nu(dx)ds + \alpha \int_0^t L_s \frac{\mathcal{E}_s(-X)}{1-\mathcal{E}_s(-X)} ds \tag{20}$$

which, together with (14)-(15), implies that L_t satisfies (11).

It is evident that $L \in \mathcal{V}$, since

$$0 \le L_t \le E(L_\infty|F_t)$$

and the family $(E(L_\infty|F_\tau), \tau \in \mathcal{T}))$ is uniformly integrable.

The proof of the uniqueness. Let Y_t be a solution of (11)-(12) from the class $\mathcal{V}$. Then it follows from (11) (after tedious application of the Itô formula) that the process

$$M_t = Y_t/(1-\mathcal{E}_t(-X)) \tag{21}$$

is a local martingale. Since X is a subordinator and $0 \le \Delta X_t < 1$ for all $t \ge 0$ we have that

$$\frac{1}{1-\mathcal{E}_t(-X)} = \frac{1}{1-e^{-\alpha t}\Pi_{s\le t}(1-\Delta X_s)} \le \frac{1}{1-e^{-\alpha t}}.$$

Therefore, from (21)

$$|M_t| \le \frac{|Y_t|}{1-e^{-\alpha t}}$$

and since $Y \in \mathcal{V}$, the process $(M_t, t\ge r)$ will be a uniformly integrable martingale for any $r>0$. This implies that $M=(M_t, t\ge r)$ can be represented as $M_t = E(\eta|F_t)$ for some $\mathcal{F}$ measurable integrable random variable η, hence

$$Y_t/(1-\mathcal{E}_t(-X)) = E(\eta|F_t), \quad t \ge r. \tag{22}$$

By the boundary condition (12) and the Levy theorem, passing to the limit as $t \to \infty$ in (22) we obtain that

$$\eta = Y_\infty = L_\infty,$$

which by arbitrariness of $r > 0$ and right-continuity of Y and M_t implies that $Y_t = L_t = E(L_\infty|F_t)[1 - \mathcal{E}_t(-X)]$ for any $t \geq 0$.

Remark 1. If X_t is a deterministic subordinator, i.e. if $X_t = \alpha t, \alpha > 0$, then equation (11) coincides with equation (6) from the introduction.

Remark 2. Note that in this model the solution L_t is no more an increasing process, but the expectation EL_t is an increasing function. Indeed, from the Itô formula

$$L_t = E(L_\infty|F_t^W)[1 - \mathcal{E}_t(-X)] = \tag{23}$$

$$\int_0^t E(L_\infty|F_s)\mathcal{E}_{s-}(-X)dX_s + \text{martingale}.$$

Since X is a Levy process with positive jumps, $X_t - EX_t$ is a martingale and EX_t is increasing. Therefore, it follows from (23) that

$$EL_t = E\int_0^t E(L_\infty|F_s)\mathcal{E}_{s-}(-X)dEX_s = \int_0^t EL_\infty\mathcal{E}_{s-}(-X)dEX_s,$$

which implies that EL_t is an increasing function.

Now suppose that the extreme size L_∞ is an integrable F_∞^W-measurable random variable. So, we assume that two sources of randomness, the random individual variability (related with L_∞) and the environmental randomness (related with the process X_t), are independent, which is natural to assume.

Under these conditions

$$E(L_\infty|F_t) = E(L_\infty|F_t^W),$$

$$L_t = E(L_\infty|F_t^W)[1 - \mathcal{E}_t(-X)]. \tag{24}$$

and the BSDE for the process L_t will be essentially simpler.

Denote by m the average size of jump of the process X

$$m \equiv \int_{R_+} x\nu(dx).$$

Since $0 < \Delta X_t < 1, t \geq 0$ and $\nu(dx)$ is the Levy measure

$$\int_{R_+} x^2 \nu(dx) < \infty,$$

hence m is finite.

Corollary 1. Let X be a Levy process with increasing paths and let $\Delta X_t < 1$ for all $t \geq 0$. Assume that L_∞ is an integrable F^W_∞-measurable random variable and the processes W and X are independent.

Then the process

$$L_t = E(L_\infty | F^W_t)[1 - \mathcal{E}_t(-X)] \quad t \geq 0$$

is the unique solution of the BSDE

$$Y_t = (\alpha + m) \int_0^t Y_s \frac{\mathcal{E}_s(-X)}{(1 - \mathcal{E}_s(-X))} ds + \int_0^t Z_s dW_s +$$

$$+ \int_0^t \int_{R_+} K(s,x)(\mu - \tilde{\mu})(dx, ds), \tag{25}$$

with the boundary condition

$$Y_\infty = \lim_{t \to \infty} Y_t = L_\infty. \tag{26}$$

Proof. It follows from (17), that the purely discontinuous martingale part of the martingale $L_t/1 - \mathcal{E}_t(-X)$ is equal to

$$\int_0^t \int_{R_+} \frac{1}{1 - \mathcal{E}_{s-}(-X)} K(s,x)(\mu - \tilde{\mu})(ds, dx) - \int_0^t \int_{R_+} L_{s-} \frac{H(s,x)}{1 - \mathcal{E}_{s-}(-X)} (\mu - \tilde{\mu})(ds, dx)$$

$$- \int_0^t \int_{R_+} K(s,x) \frac{H(s,x)}{1 - \mathcal{E}_{s-}(-X)} (\mu - \tilde{\mu})(ds, dx).$$

Since $L_t/(1 - \mathcal{E}_t(-X)) = E(L_\infty | F^W_t)$ is a continuous martingale, we have that $\nu(dx)ds$ - a.e.

$$\frac{1}{1 - \mathcal{E}_{s-}(-X)} K(s,x) - L_{s-} \frac{H(s,x)}{1 - \mathcal{E}_{s-}(-X)} - K(s,x) \frac{H(s,x)}{1 - \mathcal{E}_{s-}(-X)} = 0.$$

This implies that

$$K(s,x) = L_s \frac{H(s,x)}{1 - H(s,x)}, \qquad \nu(dx)ds - \ a.e. \tag{27}$$

and substituting this expression in (20), we obtain that

$$A_t = \int_0^t \int_{R_+} L_s \frac{H(s,x)}{1-H(s,x)} \nu(dx) ds + \alpha \int_0^t L_s \frac{\mathcal{E}_s(-X)}{1-\mathcal{E}_s(-X)} ds. \tag{28}$$

By definition of H

$$\frac{H(s,x)}{1-H(s,x)} = \frac{x\mathcal{E}(-X)}{1-\mathcal{E}(-X)},$$

therefore

$$A_t = \int_0^t \int_{R_+} L_s \frac{x\mathcal{E}_s(-X)}{1-\mathcal{E}_s(-X)} \nu(dx) ds + \alpha \int_0^t L_s \frac{\mathcal{E}_s(-X)}{1-\mathcal{E}_s(-X)} ds =$$

$$= (\alpha + m) \int_0^t L_s \frac{\mathcal{E}_s(-X)}{1-\mathcal{E}_s(-X)} ds$$

which implies that L_t satisfies (25).

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