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ON A GENERALIZATION OF KHINCHIN'S THEOREM

V. BERIKASHVILI, G. GIORGOBIANI, V. KVARATSKHELIA

ABSTRACT. A generalization of Khinchin's theorem for weakly correlated random elements with values in Banach spaces $l_p, 1 \leq p < \infty$ is presented without proof.

The purpose of this paper is to generalize the following Khinchin's theorem, which was published in 1928 in the journal of the French Academy of Sciences [1]. The concepts and background information about probability distributions in infinite-dimensional spaces, necessary for further discussion, can be found in [2].

Let $\xi_1, \xi_2, \ldots, \xi_n, \ldots$ be a sequence of real random variables, defined on the probability space $(\Omega, \mathfrak{F}, \mathbb{P})$ with finite mathematical expectations $\mathbb{E}\,\xi_n < \infty$; denote $S_n = \sum_{i=1}^{n} \xi_i, n = 1, 2, \ldots$. We say that the given sequence of random variables satisfies the *Law of Large Numbers* (*LLN*), if the sequence $\{\frac{S_n - \mathbb{E}\,S_n}{n}\}$ converges in probability to zero as $n \to \infty$, i.e. for every $\varepsilon > 0$

$$\lim_{n\to\infty} \mathbb{P}\left[\left|\frac{S_n - \mathbb{E}\,S_n}{n}\right| > \varepsilon\right] = 0.$$

Theorem 1. (A.Y. KHINCHIN, 1928). *Let $\xi_1, \xi_2, \ldots, \xi_n, \ldots$ be a sequence of random variables such that for any positive integer n, ξ_n has a finite mathematical expectation and variance σ_n^2. Furthemore, let g be a nonnegative function, defined on the set of nonnegative integers such that for the correlation coefficients ϱ_{mn} of ξ_m and ξ_n the following inequalities hold*

$$|\varrho_{mn}| \leq g(|m-n|), \qquad m, n = 1, 2, \ldots.$$

If

$$\lim_{n\to\infty} \frac{1}{n^2}\left(\sum_{i=0}^{n-1} g(i)\right)\left(\sum_{i=1}^{n} \sigma_i^2\right) = 0, \tag{1}$$

then the sequence $\xi_1, \xi_2, \ldots, \xi_n, \ldots$ satisfies the LLN.

In this paper [1] A. Khinchin introduced the well-established term "Strong Law of Large Numbers" (*SLLN*) and proved that the theorem formulated above provides a sufficient condition for the fulfillment of the *SLLN* if condition (1) is replaced by the condition

$$\lim_{n\to\infty} \frac{1}{n^{2-\delta}}\left(\sum_{i=0}^{n-1} g(i)\right)\left(\sum_{i=1}^{n} \sigma_i^2\right) = 0$$

for some $\delta > 0$.

Recall some general notions. Let X be a separable Banach space with a norm $\|\cdot\|$, X^* be its dual, $\langle x^*, x\rangle$ be a value of the functional $x^* \in X^*$ at the point $x \in X$, $(\Omega, \mathfrak{F}, \mathbb{P})$ be a fixed probability space. Denote by $\mathfrak{B}(X)$ the Borel σ-algebra in X. A map $\xi : \Omega \to X$ is called *a random element* with values in X if $\xi^{-1}\{\mathfrak{B}(X)\} \subset \mathfrak{F}$.

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It is said that a random element ξ with values in X has *a weak p-order*, $p > 0$, if $\mathbb{E}\,|\langle x^*, \xi\rangle|^p < \infty$ for every $x^* \in X^*$. If a random element ξ has a weak p-order, $p > 1$, then the *expectation* $\mathbb{E}\,\xi$ exists and is defined as the *Pettis integral* of the random element ξ. Without loss of generality we assume that all random elements considered below are centered (that is, $\mathbb{E}\,\xi$=0). For a random element ξ with the weak second order *covariance operator* $R_\xi : X^* \to X$ is defined as follows:

$$\langle x^*, R_\xi x^*\rangle = \mathbb{E}\,\langle x^*, \xi\rangle^2, \qquad x^* \in X^*.$$

It is easy to see that R_ξ is a nonnegative, symmetric and linear continuous operator. For any symmetric nonnegative operator $R : X^* \to X$ there is a Hilbert space H and a linear continuous operator $A : X^* \to H$ such that $R = A^*A$; the operator A is uniquely determined up to isometry (see [2], Factorization Lemma, p. 123).

Random element $\xi : \Omega \to X$ is called *Gaussian* if $\langle x^*, \xi\rangle$ is a Gaussian random variable for any $x^* \in X^*$. We say that an operator $R : X^* \to X$ is a *Gaussian covariance* if there exists a Gaussian random element with values in X such that its covariance operator coincides with R.

Let $X = H$ be a Hilbert space with the inner product $(\cdot, \cdot)_H$. An operator $T : H \to H$ is called *nuclear* if it admits the representation

$$Th = \sum_{i=1}^{\infty} (a_i, h)_H b_i \qquad h \in H,$$

and for some sequences $\{a_i\}$ and $\{b_i\}$ in H with $\sum_{i=1}^{\infty} \|a_i\|_H \|b_i\|_H < \infty$.

Let $\{\varphi_k\}$ be an orthonormal basis in H. Then for the nuclear operator $T : H \to H$ the series

$$\sum_{k=1}^{\infty} (T\varphi_k, \varphi_k)_H \tag{2}$$

converges, the sum (2) does not depend on the choice of the orthonormal basis and is called the *trace* ($tr(T)$) of the operator T. If a random element ξ with values in a Hilbert space has a strong second order ($\mathbb{E}\,\|\xi\|_H^2 < \infty$) and R_ξ is its covariance operator, then it is easy to see that $\mathbb{E}\,\|\xi\|_H^2 = tr(R_\xi)$.

Let ξ and η be random elements of weak second order with values in a Banach space X (recall that $\mathbb{E}\,\xi = \mathbb{E}\,\eta = 0$). *Cross-covariance operator* $R_{\xi\eta} : X^* \to X$ of ξ and η is defined by the equality:

$$\langle x^*, R_{\xi\eta} y^*\rangle = \mathbb{E}\,\langle x^*, \xi\rangle\langle y^*, \eta\rangle, \qquad x^*, y^* \in X^*.$$

It is known that $R_{\xi\eta}$ admits the factorization [3]:

$$R_{\xi\eta} = A_\xi^* V_{\xi\eta} A_\eta, \tag{3}$$

where A_ξ (resp. A_η) is a continuous linear operator from X^* to some Hilbert space H_ξ (resp. H_η) such that $R_\xi = A_\xi^* A_\xi$ (resp. $R_\eta = A_\eta^* A_\eta$), the set $A_\xi(X^*)$ (resp. $A_\eta(X^*)$) is dense in H_ξ (resp. in H_η), and $V_{\xi\eta} : H_\eta \to H_\xi$ is a continuous linear operator and for the operator norm we have $\|V_{\xi\eta}\| \le 1$.

$V_{\xi\eta}$ is called a *correlation coefficient* and as in the one-dimensional case, is a measure of the linear dependence of the random elements [3].

To prove the main result we need the following elementary lemma, which was actually applied in [1].

Lemma 2. *Let $\alpha_i, \beta_{i-1}, \varrho_{ij}$, $i, j = 1, 2, \ldots, n$, be the sequences of nonnegative numbers and let*

$$\varrho_{ij} \le \beta_{|i-j|} \quad \text{for any} \quad i, j = 1, 2, \ldots, n.$$

Then

$$\sum_{i,j=1}^{n} \varrho_{ij}\alpha_i\alpha_j \le 2\left(\sum_{i=0}^{n-1} \beta_i\right)\left(\sum_{i=1}^{n} \alpha_i^2\right).$$

Consider the Banach space of all p-absolutely convergent sequences of real numbers $l_p, 1 \le p < \infty$, with the norm $\|\cdot\|_{l_p}$. As we know the dual space is $l_p^* = l_q$, $pq = p+q$, when $1 < p < \infty$, and $l_1^* = l_\infty$.

Let ξ be a random element in $l_p, 1 \le p < \infty$, with the covariance operator R_ξ. Let $e_k = (0, \ldots, \overset{k}{1}, 0, \ldots)$, $k = 1, 2, \ldots$, be a sequence of unit vectors in the dual space l_p^*. Recall that R_ξ is a Gaussian covariance operator if and only if (see [2], Theorem 5.6, p. 261)

$$\sum_{k=1}^{\infty} \langle e_k, R_\xi e_k\rangle^{p/2} < \infty. \tag{4}$$

Let us state the main result of this paper.

Theorem 3. *Let $\xi_1, \xi_2, \ldots, \xi_n, \ldots$ be a sequence of weak second order random elements with values in $l_p, 1 \le p < \infty$, and let the covariance operator $R_n \equiv R_{\xi_n} : l_p^* \to l_p$, satisfy the condition*

$$\sigma_n^s \equiv \sum_{k=1}^{\infty} \langle e_k, R_n e_k\rangle^{s/2} < \infty, \quad n = 1, 2, \ldots, \tag{5}$$

where $s = \min\{2, p\}$. Let, besides there exists a nonnegative function g, defined on the set of nonnegative integers such that for the correlation coefficient V_{mn} of ξ_m and ξ_n the following inequalities hold

$$\|V_{mn}\| \le g(|m-n|) \qquad \text{for any} \qquad m, n = 1, 2, \ldots.$$

If

$$\lim_{n\to\infty} \frac{1}{n^2}\left(\sum_{i=0}^{n-1} g(i)\right)\left(\sum_{i=1}^{n} \sigma_i^s\right)^{2/s} = 0, \tag{6}$$

then

$$\lim_{n\to\infty} \mathbb{E}\left\|\frac{1}{n}\sum_{i=1}^{n} \xi_i\right\|_{l_p}^{s} = 0. \tag{7}$$

In particular the sequence $\xi_1, \xi_2, \ldots, \xi_n, \ldots$ satisfies the LLN.

Remark 4. *The complete proof of Theorem 3 is published in* [4].

When $p = 2$ Theorem 3 implies the following statement.

Corollary 5. *Let $\xi_1, \xi_2, \ldots, \xi_n, \ldots$ be a sequence of strong second order random elements with values in a separable Hilbert space and let a function g satisfies the requirements of Theorem 3. If*

$$\lim_{n\to\infty} \frac{1}{n^2}\left(\sum_{i=0}^{n-1} g(i)\right)\left(\sum_{i=1}^{n} tr(R_i)\right) = 0,$$

then the sequence $\xi_1, \xi_2, \ldots, \xi_n, \ldots$ satisfies the LLN.

Corollary 6. *Let $\xi_1, \xi_2, \dots, \xi_n, \dots$ be a sequence of weak second order random elements with values in $l_p, 1 \le p \le 2$, and let the covariance operators $R_n \equiv R_{\xi_n}$ satisfy the condition*

$$\sigma_n^p \equiv \sum_{k=1}^{\infty} \langle e_k, R_n e_k \rangle^{p/2} < \infty, \quad n = 1, 2, \dots.$$

If

$$\lim_{n\to\infty} \frac{1}{n} \left(\sum_{i=1}^{n} \sigma_i^p \right)^{2/p} = 0, \tag{8}$$

then the sequence $\xi_1, \xi_2, \dots, \xi_n, \dots$ satisfies the LLN.

If the random elements are pairwise independent (or not correlated), then obviously we can assume that $\sum_{i=0}^{n-1} g(i) = 1$ for any positive integer n. Thus Theorem 3 implies the following

Corollary 7. *Let $\xi_1, \xi_2, \dots, \xi_n, \dots$ be a sequence of pairwise independent weak second order random elements with values in $l_p, 1 \le p < \infty$, and let for any positive integer n covariance operators $R_n \equiv R_{\xi_n}$ satisfy (3.2).*

If

$$\lim_{n\to\infty} \frac{1}{n^2} \left(\sum_{i=1}^{n} \sigma_i^s \right)^{2/s} = 0, \quad \textit{where} \quad s = \min\{2, p\},$$

then the sequence $\xi_1, \xi_2, \dots, \xi_n, \dots$ satisfies the LLN.

In particular, for the case of a separable Hilbert space we have

Corollary 8. *Let $\xi_1, \xi_2, \dots, \xi_n, \dots$ be a sequence of pairwise independent strong second order random elements with values in a separable Hilbert space and let*

$$\lim_{n\to\infty} \frac{1}{n^2} \sum_{i=1}^{n} tr(R_i) = 0.$$

Then the sequence $\xi_1, \xi_2, \dots, \xi_n, \dots$ satisfies the LLN.

Naturally the question arises about the validity of the main theorem of the paper in the general Banach space. Does it remain true at least in the case of Banach spaces with an unconditional basis and a finite cotype? The answer to this question is not yet known to us.

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V. Berikashvili, Muskhelishvili Institute of Computational Mathematics, Georgian Technical University, Gr. Peradze str., 4, Tbilisi 0159, Georgia.

E-mail address: v.berikashvili@gtu.ge

G. Giorgobiani, Muskhelishvili Institute of Computational Mathematics, Georgian Technical University, Gr. Peradze str., 4, Tbilisi 0159, Georgia.
E-mail address: giorgobiani.g@gtu.ge

V. Kvaratskhelia, Muskhelishvili Institute of Computational Mathematics, Georgian Technical University, Gr. Peradze str., 4, Tbilisi 0159, Georgia.
E-mail address: v.kvaratskhelia@gtu.ge