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Construction of identifying and real M-estimators in general statistical model with filtration

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Abstract

General statistical model with filtration is considered. Identifying and real M-estimators are constructed. Namely, consistent, linear, asymptotically normal estimators are founded, which are basic class of estimators in robust statistics.

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A key part in robust estimation theory play the Huber M-estimators. In general, M-estimators may be viewed as follows.

Consider a sequence of filtered statistical models

$$\mathcal{E} = \left\{(\Omega^n, \mathcal{F}^n, F^n = (\mathcal{F}_t^n),\ 0 \le t \le T,\ (Q_\theta^n,\, \theta \in \Theta \subset R_1))\right\}_{n\ge1}, \tag{1}$$

where for each $n \ge 1$ and $\theta \ne \theta'$, the probability measures Q_θ^n and $Q_{\theta'}^n$ are equivalent, $Q_\theta^n \sim Q_{\theta'}^n$, $\mathcal{F}^n = \mathcal{F}_T^n$ and $T > 0$ is a number, σ-algebra $\mathcal{F}_n$ is completed and filtration F^n satisfies the usual conditions w.r.t. Q_θ^n for some, and hence, for each θ.

Let for each $\theta \in \Theta$ and $n \geq 1$ the process $(L_n(\theta, t),\ 0 \leq t \leq T)$ be a local (square integrable) Q_θ^n-martingale.

Denote $L_n(\theta) = L_n(\theta, t)|_{t=T}$ and consider stochastic equation (with respect to parameter θ)

$$L_n(\theta) = L_n(\theta, \omega) = 0, \quad n \geq 1. \tag{2}$$

A sequence $\{T_n(\omega),\ \omega \in \Omega^n\}_{n\geq 1}$ of $\mathcal{F}^n$-measurable roots of these equations $\big($i.e., for each $n \geq 1$, $T_n(\omega)$ is a random variable defined on $(\Omega^n, \mathcal{F}^n)$ with values Θ, and such that

$$L_n(T_n(\omega), \omega) = 0\,\big) \tag{3}$$

is called a generalized M-estimator.

Notice that the equality (3) may be satisfied only asymptotically (in some sense, see, e.g., Theorem 1 below).

The proof of assertions concerning the asymptotic behaviour of M-estimators as solutions of equation (2) is carried out in two steps: firstly, the asymptotic properties are established for the left-hand side of equation (2); secondly, the asymptotic properties of the estimators (considered as implicit functions) are obtained by linearization. In this way one may construct consistent, linear, asymptotically normal estimators, which are asymptotically equivalent of M-estimators (see, e.g., (15) below). Class of such estimators is a basic class of estimators in robust estimation theory (see, e.g., [1, 2, 3]).

1 Local limiting behaviour of roots

Given a sequence of statistical models (1), and let $\{c_n(\theta)\}_{n\geq 1}$, $c_n(\theta) > 0$, $\theta \in \Theta$ be a normalizing deterministic sequence.

Consider the sequence of random variables $\{L_n(\theta)\}_{n\geq 1} = \{L_n(\theta, \omega),\ \omega \in \Omega^n\}_{n\geq 1}$ depending on the parameter $\theta \in \Theta$.

Remark 1. We shall use the following abbreviation

$$Q_\theta^n\text{-}\lim_{n\to\infty} \xi_n = K,$$

where $\xi = \{\xi_n\}_{n\geq 1}$ is a sequence of random variables defined for each n on Ω^n and K is a real number, if $\forall \rho > 0$,

$$\lim_{n\to\infty} Q_\theta^n\{\omega \in \Omega^n : |\xi_n(\omega) - K| > \rho\} = 0.$$

Theorem 1. *Let the following conditions hold:*

a) *for each* $\theta \in \Theta$, $\lim_{n\to\infty} c_n(\theta) = 0$;

b) *for each* $n \geq 1$, *the mapping* $\theta \rightsquigarrow L_n(\theta)$ *is continuously differentiable in* θ Q_θ^n*-a.s.,* $(\dot{L}_n(\theta) := \frac{\partial}{\partial\theta} L_n(\theta))$;

c) *for each* $\theta \in \Theta$, *there exists a function* $\Delta_Q(\theta, y)$, $\theta, y \in \Theta$, *such that*

$$Q_\theta^n\text{-}\lim_{n\to\infty} c_n^2(\theta) L_n(y) = \Delta_Q(\theta, y) \tag{4}$$

and the equation

$$\Delta_Q(\theta, y) = 0$$

with respect to the variable y *has the unique solution* $\theta^* = b^Q(\theta)$;

d) $Q_\theta^n\text{-}\lim_{n\to\infty} c_n^2(\theta)\dot{L}_n(\theta^*) = -\gamma_Q(\theta)$, *where* $\gamma_Q(\theta)$ *is a positive number for each* $\theta \in \Theta$;

e) $\lim_{r\to 0} \limsup_{n\to\infty} Q_\theta^n\{\sup_{\{y:|y-\theta^*|\leq r\}} c_n^2(\theta)|\dot{L}_n(y) - \dot{L}_n(\theta^*)| > \rho\} = 0$ *for each* $\rho > 0$.

Then for each $\theta \in \Theta$ *there exists a sequence of random variables* $T = \{T_n\}_{n\geq 1}$ *taking the values in* Θ *such that*

I. $\lim_{n\to\infty} Q_\theta^n\{L_n(T_n) = 0\} = 1$;

II. $Q_\theta^n\text{-}\lim_{n\to\infty} T_n = \theta^*$;

III. *if* $\{\widetilde{T}_n\}_{n\geq 1}$ *is another sequence with properties I and II, then*

$$\lim_{n\to\infty} Q_\theta^n\{T_n = \widetilde{T}_n\} = 1.$$

If, in addition,

f) *the sequence of distributions* $\{\mathcal{L}\{c_n(\theta)L_n(\theta^*) \mid Q_\theta^n\}\}_{n\geq 1}$ *weakly converges to a certain distribution* Φ,

then

IV. (i) $\mathcal{L}\{\gamma_Q(\theta)c_n^{-1}(\theta)(T_n - \theta^*) \mid Q_\theta^n\} \xrightarrow{w} \Phi$,

(ii) $c_n^{-1}(\theta)(T_n - \theta^*) = \dfrac{c_n^{-1}(\theta)L_n(\theta^*)}{\gamma_Q(\theta)} + R_n(\theta), \quad R_n(\theta) \xrightarrow{Q_\theta^n} 0.$

Proof. 1. By the Taylor formula we have

$$L_n(y) = L_n(\theta^*) + \dot{L}_n(\theta^*)(y - \theta^*) + [\dot{L}_n(\bar{\theta}) - \dot{L}_n(\theta^*)](y - \theta^*),$$

where $\bar{\theta} = \theta^* + \alpha(\theta^*)(y - \theta^*)$, $\alpha(\theta^*) \in [0, 1]$ and the point $\bar{\theta}$ is chosen so that $\bar{\theta} \in \mathcal{F}^n$ ($\xi \in \mathcal{F}$ means that r.v. ξ is $\mathcal{F}$-measurable).

From this we get

$$c_n^2(\theta)L_n(y) = c_n^2(\theta)L_n(\theta^*) - \gamma_Q(\theta)(y - \theta^*) + \varepsilon_n(\bar{\theta}, \theta^*)(y - \theta^*), \tag{5}$$

where $\varepsilon_n(y, \theta^*) \in \mathcal{F}^n$,

$$\varepsilon_n(y, \theta^*) = c_n^2(\theta)[\dot{L}_n(y) - \dot{L}_n(\theta^*)] + [c_n^2(\theta)\dot{L}_n(\theta^*) + \gamma_Q(\theta)], \quad y \in \Theta.$$

Evidently, conditions d) and e) ensure that

$$\lim_{r \to 0} \limsup_{n \to \infty} Q_\theta^n \Big\{ \sup_{\{y : |y - \theta^*| \le r\}} |\varepsilon_n(y, \theta^*)| > \rho \Big\} = 0 \tag{6}$$

for each $\rho > 0$.

2. We now show that there exists a family $\{\Omega_\theta(n, r) : n \ge 1, \ r > 0, \ \theta \in \Theta\}$ with properties

1) $\Omega_\theta(n, r) \in \mathcal{F}^n$,
2) $\lim\limits_{r \to 0} \limsup\limits_{n \to \infty} Q_\theta^n\{\Omega_\theta(n, r)\} = 1$,

and for any $r > 0$, $n \ge 1$ and $\omega \in \Omega_\theta(n, r)$ the equation

$$L_n(y) = 0$$

has the unique solution T_n in the segment $|y - \theta^*| \le r$.

Expansion (5) yields

$$c_n^2(\theta)L_n(\theta^* + u)u = c_n^2(\theta)L_n(\theta^*)u - u^2\gamma_Q(\theta) + u^2\varepsilon_n(\bar{\theta}, \theta^*). \tag{7}$$

For any $\theta \in \Theta$, $n \ge 1$ and $r > 0$ define

$$\Omega_\theta(n, r) = \Big\{ \omega \in \Omega^n : \ |c_n^2(\theta)L_n(\theta^*)| \le \frac{\gamma_Q(\theta)r}{2}, \quad \sup_{\{y : |y - \theta^*| \le r\}} |\varepsilon_n(y, \theta^*)| < \frac{\gamma_Q(\theta)}{2} \Big\}.$$

Obviously, $\Omega_\theta(n,r) \in \mathcal{F}^n$. Hence, if $\omega \in \Omega_\theta(r,n)$, then from equality (7) we get $L_n(\theta^* + u)u < 0$ for $|u| = r$.

Since the mapping $u \rightsquigarrow L_n(\theta^* + u)$ is continuous with respect to u, the equation $L_n(\theta^* + u) = 0$ for $|u| \le r$ has at least one solution $u_n(\theta^*)$ with $|u_n(\theta^*)| \le r$.

It can be easily seen that if $\omega \in \Omega_\theta(n,r)$ and $|u| \le r$, then $\dot{L}_n(\theta^* + u) < 0$.

On the other hand, for $\omega \in \Omega_\theta(n,r)$ and $|u| \le r$,

$$
\begin{aligned}
&L_n(\theta^* + u, \omega) - L_n(\theta^* + u_n(\theta), \omega) \\
&\qquad = \int_0^1 \frac{\partial}{\partial \alpha} \left[L_n((\theta^* + u_n(\theta^*)) + \alpha(u - u_n(\theta^*)), \omega) \right] d\alpha.
\end{aligned}
$$

Consequently,

$$
L_n(\theta^* + u, \omega) = \int_0^1 \dot{L}(\theta^* + u_n(\theta^*) + \alpha(u - u_n(\theta^*)), \omega)(u - u_n(\theta^*))\, d\alpha
$$

and

$$
\begin{aligned}
&L_n(\theta^* + u, \omega)(u - u_n(\theta^*)) \\
&\qquad = \int_0^1 \dot{L}(\theta^* + u_n(\theta^*) + \alpha(u - u_n(\theta^*)), \omega)(u - u_n(\theta^*))^2\, d\alpha < 0,
\end{aligned}
$$

provided $u \ne u_n(\theta^*)$. Hence $L_n(\theta^* + u, \omega) \ne 0$ for $|u| \le r$, $u \ne u_n(\theta^*)$. By the construction of the set $\Omega_\theta(n,r)$ and due to conditions c), d) and e) it is easily seen that 2) is true as well.

3. Now we construct the sequence $T = \{T_n\}_{n \ge 1}$ with properties I, II and III. Define

$$
\Omega_n^\theta := \bigcup_{k > 0} \Omega_\theta(n, k^{-1}).
$$

Evidently, $\Omega_n^\theta \in \mathcal{F}^n$. Let $\omega \in \Omega_n^\theta$. Then from the previous statement it follows that there exists a number $k(\omega) > 0$ such that the equation $L_n(y) = 0$ has the unique solution $\widetilde{T}_n(\omega)$ in the segment $|y - \theta^*| \le (k(\omega))^{-1}$ with the mapping $\omega \rightsquigarrow \widetilde{T}_n(\omega)$ which is $\Omega_n^\theta \cap \mathcal{F}^n$-measurable.

Put

$$
T_n(\omega) = \begin{cases} \widetilde{T}_n(\omega) & \text{if } \omega \in \Omega_n^\theta, \\ \theta_0 & \text{if } \omega \ne \Omega_n^\theta, \end{cases}
$$

where θ_0 is a point in Θ.

It is easily seen that, by construction, T_n possesses properties I, II and III.

4. Finally, we prove assertion IV. By expansion (5), we have

$$\begin{aligned} &|c_n(\theta)L_n(T_n) - c_n(\theta)L_n(\theta^*) - \gamma_Q(\theta)c_n^{-1}(\theta)(T_n - \theta^*)| \\ &\quad \le |\varepsilon_n(\bar{T}, \theta^*)\gamma_Q^{-1}(\theta)|\,|\gamma_Q(\theta)c_n^{-1}(\theta)(T_n - \theta^*)| \end{aligned} \tag{8}$$

and $\limsup_{n\to\infty} Q_\theta^n\{|\varepsilon_n(\bar{T}_n, \theta^*)| \ge \rho\} = 0$, $\forall \rho > 0$, which follows directly from the relation

$$\{|\bar{T}_n - \theta^*| \le r\} \cap \Big\{ \sup_{\{y:|y-\theta^*|\le r\}} |\varepsilon_n(y, \theta^*)| < \rho \Big\} \subset \{|\varepsilon_n(\bar{T}_n, \theta^*)| < \rho\}.$$

Denote $X_n := c_n(\theta)(L_n(T_n) - L_n(\theta^*))$, $Y_n := \gamma_Q(\theta)c_n^{-1}(\theta)(T_n - \theta^*)$ and $Z_n := |\varepsilon_n(\bar{T}_n, \theta^*)\gamma_Q^{-1}|$. Then inequality (8) takes the form

$$|X_n - Y_n| \le Z_n|Y_n|.$$

It is well-known that if X_n converges weakly to X ($X_n \xrightarrow{w} X$) and $Z_n \xrightarrow{P} 0$, then $Y_n \xrightarrow{w} X$. Thus we get

$$\lim_{n\to\infty} \mathcal{L}\{\gamma_Q(\theta)c_n^{-1}(\theta)(T_n - \theta^*) \mid Q_\theta^n\} = \lim_{n\to\infty} \mathcal{L}\{c_n(\theta)L_n(\theta^*) \mid Q_\theta^n\}.$$

Assertion (i) is proved. The proof of assertion (ii) easily follows from (i) and inequality (8). □

2 Global limiting behaviour of roots

We use the objects introduced in the previous section.

Assume $\Theta = [a, b]$. Furthermore, for convenience, put $a = -\infty$ and $b = +\infty$.

For every θ we consider the set

$$\begin{aligned} S_\theta = \Big\{ \widehat{T} = \{\widehat{T}_n\}_{n\ge1} : \text{for each } n \ge 1,\ \widehat{T}_n \in \mathcal{F}^n \text{ and } \\ Q_\theta^n\text{-}\lim_{n\to\infty} c_n^2(\theta)L_n(\widehat{T}_n) = 0 \Big\}. \end{aligned}$$

Theorem 2. *Let the following conditions* $(\sup c)$ *hold:*

$(\sup c)_1$ *the function* $\Delta_Q(\theta, y)$ *is* y*-continuous for every* θ*;*

$(\sup c)_2$ *for any* K, $0 < K < \infty$, *and* $\rho > 0$,

$$\lim_{n\to\infty} Q_\theta^n \Big\{ \sup_{|y|\le K} |c_n^2(\theta) L_n(y) - \Delta_Q(\theta, y)| > \rho \Big\} = 0.$$

Then

I. *The following alternative holds: if* $\widehat{T} \in S_\theta$*, then either*

$$Q_\theta^n\text{-}\lim_{n\to\infty} \widehat{T}_n = \theta^* = b^Q(\theta), \tag{9}$$

or

$$\overline{\lim_{n\to\infty}}\, Q_\theta^n\{|\widehat{T}_n| > K\} > 0 \tag{10}$$

for any K, $0 < K < \infty$.

II. *If, in addition, the condition*

(c^+)
$$\lim_{|y|\to\infty} |\Delta_Q(\theta, y)| = K(\theta) > 0$$

holds and

$$\lim_{n\to\infty} Q_\theta^n \Big\{ \sup_{-\infty<y<+\infty} |c_n^2(\theta) L_n(y) - \Delta_Q(\theta, y)| > \rho \Big\} = 0$$

for any $\rho > 0$*, then* (9) *is valid.*

Proof. Let $\widehat{T} = \{\widehat{T}_n\}_{n\ge1} \in S_\theta$ and suppose that inequality (10) is not satisfied. Then there is a number $K_0 > 0$ such that

$$\lim_{n\to\infty} Q_\theta^n\{|\widehat{T}_n| > K_0\} = 0.$$

Therefore,

$$\begin{aligned} &Q_\theta^n\big\{|c_n^2(\theta) L_n(\widehat{T}_n) - \Delta_Q(\theta, \widehat{T}_n)| > \rho\big\} \\ &\quad \le Q_\theta^n\big\{|\widehat{T}|_n > K_0\big\} + Q_\theta^n\Big\{|c_n^2(\theta) L_n(\widehat{T}_n) - \Delta_Q(\theta, \widehat{T}_n)| > \rho,\ \ |\widehat{T}_n| \le K_0\Big\} \\ &\le Q_\theta^n\big\{|\widehat{T}|_n > K_0\big\} + Q_\theta^n\Big\{ \sup_{|y|\le K_0} |c_n^2(\theta) L_n(y) - \Delta(\theta, y)| > \rho\Big\} \to 0 \ \text{ as } \ n\to\infty. \end{aligned}$$

On the other hand,

$$Q_\theta^n\text{-}\lim_{n\to\infty} c_n^2(\theta) L_n(\widehat{T}_n) = 0$$

and hence,

$$Q_\theta^n\text{-}\lim_{n\to\infty} \Delta_Q(\theta, \widehat{T}_n) = 0. \tag{11}$$

Assume now that equality (9) fails too. Then one can choose $\varepsilon > 0$ such that

$$\overline{\lim_{n\to\infty}}\, Q_\theta^n\big\{|\widehat{T}_n - b^Q(\theta)| > \varepsilon\big\} > 0.$$

By the condition $(\sup c)_1$,

$$\Delta(\varepsilon) = \inf_{\{y : |y - b^Q(\theta)| > \varepsilon,\, |y| \le K_0\}} |\Delta_Q(\theta, y)| > 0,$$

whence

$$\begin{aligned}
&\overline{\lim_{n\to\infty}}\, Q_\theta^n\big\{|\Delta_Q(\theta, \widehat{T}_n)| > \Delta(\varepsilon)\big\} \\
&\qquad \ge \overline{\lim_{n\to\infty}}\, Q_\theta^n\big\{|\Delta_Q(\theta, \widehat{T}_n)| > \Delta(\varepsilon),\ \ |\widehat{T}_n| \le K_0\big\} \\
&\qquad \ge \overline{\lim_{n\to\infty}}\, Q_\theta^n\big\{|\widehat{T}_n - b^Q(\theta)| > \varepsilon,\ \ |\widehat{T}_n| \le K_0\big\} > 0,
\end{aligned}$$

which contradicts equality (11).

In order to prove the second assertion of theorem, it is sufficient to note that under the condition (c^+)

$$\inf_{\{y : |y - b^Q(\theta)| \ge \varepsilon\}} |\Delta_Q(\theta, y) > 0$$

and to repeat the previous arguments. □

Suppose that the conditions of Theorem 1 are satisfied.

For every $n \ge 1$, consider the set

$$A_n = \{\omega \in \Omega^n : \text{ the equation } L_n(y, \omega) = 0 \text{ has at least one solution}\}.$$

Note that $A_n \in \mathcal{F}^n$. Indeed, recall that the σ-algebra $\mathcal{F}^n$ is complete, $L_n(y, \cdot) \in \mathcal{F}^n$ for each fixed y and $L_n(\cdot, \omega)$ is a.s. continuous. Hence, the mapping $(y, \omega) \rightsquigarrow L_n(y, \omega)$ is measurable and $B_n := \{(y, \omega) : L_n(y, \omega) = 0\} \in \mathcal{B}(R_1) \times \mathcal{F}^n$. But $A_n = \Pi_{\Omega^n}(B_n)$, where $\Pi_{\Omega^n}(\cdot)$ is a projection operator. Thus $A_n \in \mathcal{F}^n$.

Evidently, for any θ, we have $\Omega_n^\theta \subset A_n$, where the set Ω_n^θ is defined in item 3 of the proof of Theorem 1.

Since under the conditions of Theorem 1, $Q_\theta^n\{\Omega_n^\theta\} \to 1$, for any θ we have

$$\lim_{n\to\infty} Q_\theta^n\{A_n\} = 1.$$

For each $n \geq 1$, introduce the sets:

$$S_n = \{\widetilde{T}_n : \ \widetilde{T}_n \text{ is } \mathcal{F}^n\text{-measurable; } L_n(\widetilde{T}_n) = 0 \text{ if } \omega \in A_n;\ \widetilde{T}_n = \theta_0 \text{ if } \omega \notin A_n\},$$

where θ_0 is a real number.

Now, put the set of estimators

$$S_{sol} = \{\widetilde{T} = \big\{\widetilde{T}_n\}_{n\geq 1} : \forall n \geq 1,\ \widetilde{T}_n \in S_n\big\}.$$

Corollary 1. *If along with the conditions of Theorem 1 the conditions* $(\sup c)$ *are satisfied for any θ, then there exists an estimator $T^* = \{T_n^*\}_{n\geq 1} \in S_{sol}$ such that*

$$Q_\theta^n\text{-}\lim_{n\to\infty} T_n^* = b^Q(\theta) \tag{12}$$

for any θ.

If, moreover, for any θ the condition (c^+) *is satisfied, then any estimator $\widetilde{T} \in S_{sol}$ has property* (12).

Proof. It is sufficient to construct an estimator $T^* = \{T_n^*\}_{n\geq 1}$ for which (10) fails for each θ.

For any $n \geq 1$ and $\varepsilon > 0$, there exists $T_n^* \in S_n$ such that

$$|T_n^*| \leq \operatorname*{ess\,inf}_{\widetilde{T}_n\in S_n} |\widetilde{T}_n| + \varepsilon.$$

By virtue of Theorem 1, for any θ there exists a sequence $\widehat{T}(\theta) = \{\widehat{T}_n(\theta)\}_{n\geq 1}$ such that

$$\lim_{n\to\infty} Q_\theta^n\{L_n(\widehat{T}_n(\theta)) = 0\} = 1 \tag{13}$$

and

$$Q_\theta^n\text{-}\lim_{n\to\infty} \widehat{T}_n(\theta) = b^Q(\theta). \tag{14}$$

Thus, we have

$$\begin{aligned}
\overline{\lim_{n\to\infty}}\, Q_\theta^n\big\{|T_n^*| > K\big\} &\leq \overline{\lim_{n\to\infty}}\, Q_\theta^n\big\{|T_n^*| > K,\ L_n(\widehat{T}_n(\theta)) \neq 0\big\} \\
&\quad + \overline{\lim_{n\to\infty}}\, Q_\theta^n\big\{|T_n^*| > K,\ L_n(\widehat{T}_n(\theta)) = 0\big\} \\
&\leq \overline{\lim_{n\to\infty}}\, Q_\theta^n\big\{L_n(\widehat{T}_n(\theta)) \neq 0\big\} + \overline{\lim_{n\to\infty}}\, Q_\theta^n\big\{|\widehat{T}_n(\theta)| + \varepsilon > K\big\}.
\end{aligned}$$

The first and the second terms on the right-hand side converge to zero by virtue of equalities (13) and (14). □

Remark 2. If the conditions of Corollary 1 are satisfied, then by virtue of Theorem 1, IV (ii), there exists an estimator $T = \{T_n\}_{n\geq 1}$ such that

$$T_n = \theta^* + \frac{L_n(\theta^*)}{\gamma_Q(\theta)} + R_n(\theta), \tag{15}$$
$$c_n^{-1}(\theta)R_n(\theta) \xrightarrow{Q_\theta^n} 0.$$

If $\theta^* = b^Q(\theta) = \theta$ and the distribution Φ from Theorem 1, f), is Gaussian, then we obtain a consistent, linear, asymptotically normal estimator.

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