DOI 10.48614/bara.7.2022.6041

Number of Unordered Samples of Integers With a Given Sum

Tsotne Kutalia

Cybernetics Institute of Georgian Technical University.

Abstract

There is an analytic formula counting the number of ordered samples of $\boldsymbol{N}$ non-negative integers making up a given sum. In this paper we study the number of unordered samples of $\boldsymbol{N}$ non-negative integers with a given sum. We produce a closed form solution for $\boldsymbol{N} = \mathbf{3}$ non-negative integers.

Keywords: Combinatorics, Number Theory, Graph Theory

1 Introduction

A typical approach to finding the total number of ordered samples $(a_1+...+a_N)$ of N non-negative integers making up a sum of $n(n \geq N, a_1 + ...a_N = n)$ is to take n ones $1 + 1 + ... + 1$ (n times) and put $N - 1$ separator bars in the sequence. The total umber of arrangements of bars and ones can be viewed as the total number of ordered arrangements of $N - 1$ zeros and n ones which obviously is C^n_{n+N-1}. See [1].
However, the same problem gets complicated for unordered samples. There is a known recursion in ([2]) which is defined as

$$f_N(n) = f_{N-1}(n) + f_N(n - N) \tag{1}$$

In the text that follows, we obtain a precise formula for $N = 3$ and $n \geq N$ to be

$$f_3(n) = I_{\{\{n\}_3=0\}}\left[\frac{(n+3)(n+6)}{18} + I_{\{\{n\}_2=0\}}\frac{n^2}{36} + I_{\{\{n\}_2\neq 0\}}\frac{(n-3)(n+3)}{36}\right] + I_{\{\{n\}_3=1\}}\left[\frac{(n+2)(n+5)}{18} + I_{\{\{n\}_2=0\}}\frac{(n+2)^2}{36} + I_{\{\{n\}_2\neq 0\}}\frac{(n-1)(n+5)}{36}\right] + I_{\{\{n\}_3=2\}}\left[\frac{(n+1)(n+4)}{18} + I_{\{\{n\}_2=0)\}}\frac{(n+4)^2}{36} + I_{\{\{n\}_2\neq 0\}}\frac{(n+1)(n+7)}{36}\right] \quad (2)$$

which reduces to

$$f_3(n) = \frac{(n+3-\{n\}_3)(n+6-\{n\}_3)}{18} + \frac{(n+2(\{n\}_3)^2)-(3\{n\}_3)^2}{36} \quad (3)$$

where $\{n\}_k$ denotes a modulo operator giving a remainder for division of n over k.

2 Graphical Representation of Partitions, N=3

Let us denote by $f_N(n)$ the function counting the number of unordered samples of N non-negative integers $[a_1, ..., a_N]$ such that $a_1 + ... + a_N = n$. Let by convention $f_0(n) = 1$ for all n. Obviously $f_1(n) = 1$ for all n as well. It can easily be checked that for even $n, f_2(n) = \frac{n+2}{2}$ and for odd n we have $f_2(n) = \frac{n+1}{2}$. We can thus define $f_2(n)$ with the indicator functions as

$$f_2(n) = I_{\{n \ mod \ 2=0\}}\frac{n+2}{2} + I_{\{n \ mod \ 2=1\}}\frac{n+1}{2} \quad (4)$$

For $N = 3$, we take the sum of N ones and partition the sum of series with 2 separator bars. This can best be illustrated through an example. For $n = 3$ we have the following arrangements of 2 separator bars

$$|| \ 1+1+1, \qquad | \ 1 \ | +1+1, \qquad 1 \ | +1 \ | +1 \quad (5)$$

The first arrangement in (5) corresponds to $a_1 = 0, a_2 = 0, a_3 = 3$. The second arrangement corresponds to $a_1 = 0, a_2 = 1, a_3 = 2$ and the last one to $a_1 = 1, a_2 = 1, a_3 = 1$. So we have the followig sample $\{003, 012, 111\}$. Note that the numbers in each sample are listed in a non-decreasing order. That is why the arrangement like $| \ 1+1 \ | +1$ are ignored since that would correspond to the sample element 021 in which the numbers are not put in non-decreasing order and thus such an element already extists as 012.

We can enumerate the positions of separator bars in the series of ones as follows ${}^1 1^2 + 1^3 + 1^4$ where the superscripts mark the positions of possible placements of the separator bars. Then the sample $\{003, 012, 111\}$ can be transformed into the following sample $\{11, 12, 23\}$. In this sample, the first element 11

stands for the two bars placed at the position 1 and thus it corresponds to the first partition in (5). The element 12 corresponds to the second partition and respectively 23 is for the third partition.
We view the sample elements as the coordinates of points on the cartesian coordinate system and for convenience we reverse the numbers. So $\{11, 12, 23\}$ becomes $\{11, 21, 32\}$. The points on the coordinate system corresponding to this sample is

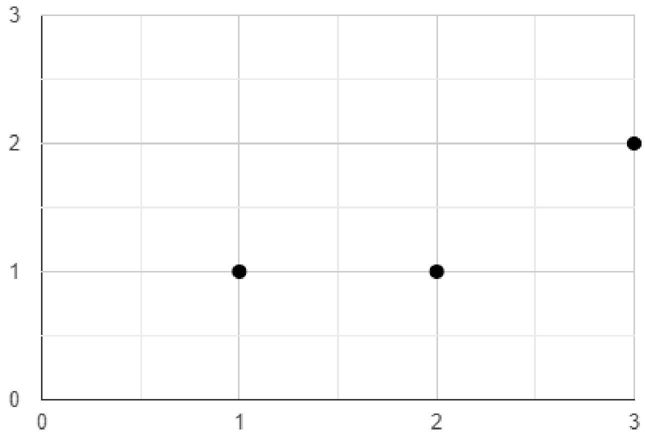

Fig. 1: $f_3(3) = 3$

Likewise, for $n = 4$ we have the following partitions identical to (5) (the corresponding samples and the reverse versions of them are given below each partition)

$$|| 1+1+1+1, \quad | 1 | +1+1+1, \quad 1 | +1 | +1+1, \quad 1 | +1 | +1+1 \quad (6)$$

Partition:	$\|\| 1+1+1+1$	$\| 1 \| +1+1+1$	$1 \| +1 \| +1+1$	$1 \| +1 \| +1+1$
Sample:	004	013	022	112
Coordinate:	11	12	13	23
Reversed:	11	21	31	32

and the corresponding plot for $f_3(4) = 4$ is on Fig.2 below.

The appendix at the end of the paper contains some of the partitions and the respective graphs. We take some of the examples here to develop the

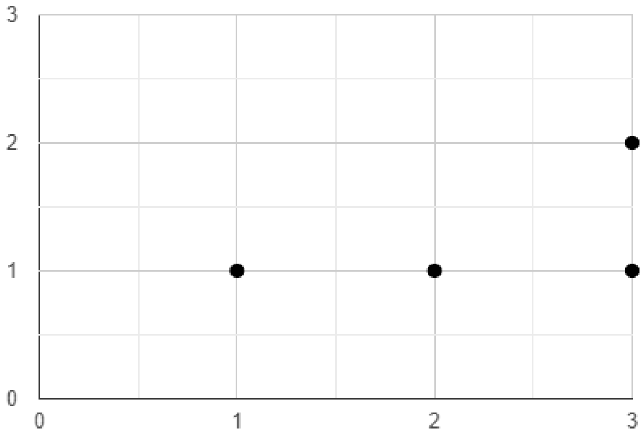

Fig. 2: $f_3(4) = 4$

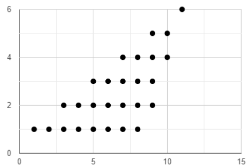

Fig. 3: $f_3(15) = 27$

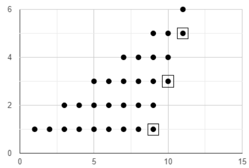

Fig. 4: $f_3(16) = 30$

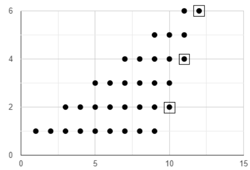

Fig. 5: $f_3(17) = 33$

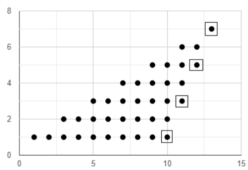

Fig. 6: $f_3(18) = 37$

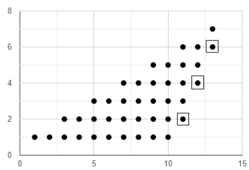

Fig. 7: $f_3(19) = 40$

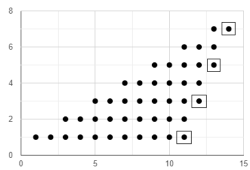

Fig. 8: $f_3(20) = 44$

formula (2). The examples for $N = 3$ are $n = 15, n = 16, n = 17, n = 18, n = 19, n = 20$.
Let us begin with $n = 15, n = 16$ and $n = 17$ on the one hand and $n = 18, n =$

$19, n = 20$ on another. The respective graphs are given in Fig. 3 to Fig. 8. There are some interesting patters emerging. In particular, we have 3 possible configurations listed below

Configuration 1, $n\ mod\ 3 = 0$: Fig.3 displays the case when $n = 15$ which is divisible by 3. On that graph there is an extreme point placed at the coordinate $(11, 6)$. This point is unique in the sense that it does not share either x or y coordinate with any other point. In general, there is a point located at the coordinate (x_0, y_0) while there is no any other point having either x_0 as x coordinate or y_0 as y coordinate. The value of y_0 coordinate can be found by

$$y_0 = 1 + \frac{n}{3} \tag{7}$$

Configuration 2, $n\ mod\ 3 = 1$: Fig. 4 displays the case when $n = 16$. The points put in squares indicate the additions to the previous graph. So as we move from Fig. 3 to Fig. 4 we have new points added on the coordinates $(11, 5)$, $(10, 3)$ and $(9, 1)$. In general, we have the points added on the coordinates $(x_0, y_0 - 1)$, $(x_0 - 1, y_0 - 3)$ and so on till the last y coordinate reaches 1. i.e. $y = 1$. The value of y_0 coordinate now is

$$y_0 = 1 + \frac{n-1}{3} \tag{8}$$

Configuration 3, $n\ mod\ 3 = 2$: Fig. 5 displays the case when $n = 17$. Again, the points in the squares indicate the additions from the previous case. In particular when moving from Fig. 4 to Fig. 5 we have the new points added on the coordinates $(12, 6)$, $(11, 4)$ and $(10, 2)$. In general, the points are added on $(x_0 + 1, y_0)$, $(x_0, y_0 - 2)$ and so on till the last point's y coodinate reaches 2. The value of y_0 for this configuration is

$$y_0 = 1 + \frac{n-2}{3} \tag{9}$$

In total we only have these 3 configurations and the cycle goes over and over again. For example, when $n = 18$, the configuration is similar to the case when $n = 15$. In general, all $n\ mod\ 3 = 0$ configurations are similar with a slight difference. When n is odd, the last added point occurs at the coordinate $y = 2$ while for even n, the additions continue till $y = 1$. Similar differences hold for cases $n\ mod\ 3 = 1$ and $n\ mod\ 3 = 2$. In particular, for odd $n - 1$, we keep adding points as described in Configuration 2 till the last point's y coordinate is $y = 1$ while for even $n - 1$, the last point added occurs at the y coordinate of $y = 2$. Similarly for odd $n - 2$ we have the last added point's y coordinate to be $y = 2$ and for even $n - 2$ we have the same coordinate to be $y = 1$.

$f_3(n)$ is simply the number of points on a corresponding plot. In order to count them we take the diagonal approach. Let us observe the counting method for all 3 configurations for the above mentioned examples.

Configuration 1: We can split the total number of points on Fig. 3 into two parts. The upper part of (and including) the main diagonal and the lower part.

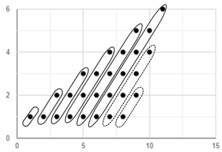

Fig. 9: $f_3(15) = 27$

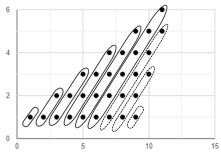

Fig. 10: $f_3(16) = 30$

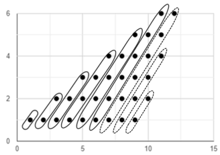

Fig. 11: $f_3(17) = 33$

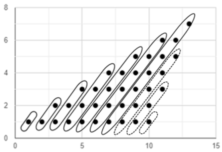

Fig. 12: $f_3(18) = 37$

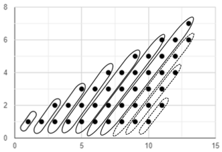

Fig. 13: $f_3(19) = 40$

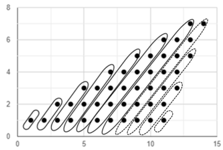

Fig. 14: $f_3(20) = 44$

We refer to the formula of the sum n terms of arithmetic series which in its more convenient form can according to [REFERENCE HERE] be written as

$$S_n = \frac{n}{2}(a_1 + a_n) \tag{10}$$

where a_1 and a_n are respectively the first and the last terms of the series.
In configuration 1, the upper part of (and including) the longest diagonal is summed as $1 + 2 + ... + (1 + \frac{n}{3})$ where the last term comes from (7). By (10) this sum is $\frac{(n+3)(n+6)}{18}$. As for the lower part of the diagonal, we have 2 variations. In particular, when n is odd (the case shown on Fig. 9) the sum of the arithmetic series with the common difference of 2 consisting of the following terms $1 + 3 + 5 + ... + (\frac{n}{3} - 1)$ which by (10) is $\frac{n^2}{36}$. On the other hand, if n is even (the case shown on Fig. 12), the sum of the arithmetic series is $2 + 4 + 6 + ... + (\frac{n}{3} - 1)$ which by (10) is $\frac{(n-3)(n+3)}{36}$. Combining these terms yields the number of unordered samples of 3 non - negative integers with a sum n when $n \ mod \ 3 = 0$ which is

$$I_{\{\{n\}_3=0\}} \left[\frac{(n+3)(n+6)}{18} + I_{\{\{n\}_2=0\}} \frac{n^2}{36} + I_{\{\{n\}_2\neq 0\}} \frac{(n-3)(n+3)}{36} \right]. \tag{11}$$

Similarly, for configuration 2, the upper part of (and including) the longest diagonal is summed as $1 + 2 + ... + (1 + \frac{n-1}{3})$ which by (10) is $\frac{(n+2)(n+5)}{18}$. The lower parts differ according to whether $n - 1$ is odd or even. For odd $n - 1$ (the case shown on Fig. 10) the sum of the arithmetic series is $2 + 4 + 6 +$

$... + \frac{n-1}{3}$ which by (10) is $\frac{(n-1)(n+5)}{18}$. For $n-1$ being even, the series becomes $1+3+5+...+\frac{n-1}{3}$ which by (10) is $\frac{(n+2)^2}{36}$. Combininig these terms yields the number of unordered samples of 3 non - negative integers with a sum n when $n \; mod \; 3 = 1$ which is

$$I_{\{\{n\}_3=1\}}\left[\frac{(n+2)(n+5)}{18} + I_{\{\{n\}_2=0\}}\frac{(n+2)^2}{36} + I_{\{\{n\}_2\neq 0\}}\frac{(n-1)(n+5)}{36}\right]. \tag{12}$$

Finally, for configuration 3, the upper part of (and including) the lonest diagonal is summed as $1+2+3+...+(1+\frac{n-3}{3})$ which by (10) is $\frac{(n+1)(n+4)}{38}$. The lower parts similarly to the previous configurations is differ according to $n-2$ being odd or even. For odd $n-2$, the sum is $2+4+...+\frac{n-1}{3}$ which by (10) is $\frac{(n+4)^2}{36}$ and for odd $n-2$ the sum $1+3+...+(1+\frac{n-3}{3})$ by (10) is $\frac{(n+1)(n+7)}{36}$. Combininig these terms yields the number of unordered samples of 3 non - negative integers with a sum n when $n \; mod \; 3 = 2$ which is

$$I_{\{\{n\}_3=2\}}\left[\frac{(n+1)(n+4)}{18} + I_{\{\{n\}_2=0\}}\frac{(n+4)^2}{36} + I_{\{\{n\}_2\neq 0\}}\frac{(n+1)(n+7)}{36}\right] \tag{13}$$

In total, $f_3(n)$ turns out to be the sum of (11), (12) and (13) which is (2) restated below

$$\begin{aligned} f_3(n) = I_{\{\{n\}_3=0\}}&\left[\frac{(n+3)(n+6)}{18} + I_{\{\{n\}_2=0\}}\frac{n^2}{36} + I_{\{\{n\}_2\neq 0\}}\frac{(n-3)(n+3)}{36}\right] + \\ I_{\{\{n\}_3=1\}}&\left[\frac{(n+2)(n+5)}{18} + I_{\{\{n\}_2=0\}}\frac{(n+2)^2}{36} + I_{\{\{n\}_2\neq 0\}}\frac{(n-1)(n+5)}{36}\right] + \\ I_{\{\{n\}_3=2\}}&\left[\frac{(n+1)(n+4)}{18} + I_{\{\{n\}_2=0)\}}\frac{(n+4)^2}{36} + I_{\{\{n\}_2\neq 0\}}\frac{(n+1)(n+7)}{36}\right] \end{aligned}$$

It is easily verified that the formula above can be reduced to (3) component by component. This is also restated below

$$f_3(n) = \frac{(n+3-\{n\}_3)(n+6-\{n\}_3)}{18} + \frac{(n+2(\{n\}_3)^2)-(3\{n\}_3)^2}{36}$$

At this point it remains to prove the formula. This is done by induction (1) part by part.

To prove that the configuration 1 part of the formula holds for any $n \geq 3$, we assume that it holds for some $n \geq 3$ and show that it also holds for $n+3$. In fact, it can easily be shown that if we put $n+3$ in place of n in (7), we obtain the following sum in place of (11)

$$
\begin{aligned}
I_{\{\{n+3\}_3=0\}}(1+2+3+...+(1+\frac{n+3}{3})+\\
I_{\{\{n+3\}_2=0\}}(1+3+5+...+(\frac{n+3}{3}-1))+\\
I_{\{\{n+3\}_2\neq0\}}(2+4+6+...+(\frac{n+3}{3}-1))).
\end{aligned} \tag{14}
$$

Applying (10) to each component yields

$$
I_{\{\{n+3\}_3=0\}}\left[\frac{6+n}{6}\frac{9+n}{3}+I_{\{\{n+3\}_2=0\}}\frac{(n+3)^2}{36}+I_{\{\{n+3\}_2\neq0\}}\frac{n(n+6)}{36}\right]. \tag{15}
$$

Similarly, for configuration 2 part of the formula, we get

$$
\begin{aligned}
I_{\{\{n+3\}_3=1\}}(1+2+3+...+(1+\frac{n+2}{3})+\\
I_{\{\{n+3\}_2=0\}}(2+4+6+...+\frac{n+2}{3})+\\
I_{\{\{n+3\}_2\neq0\}}(1+3+5+...+\frac{n+2}{3})).
\end{aligned} \tag{16}
$$

This by applying (10) becomes

$$
I_{\{\{n+3\}_3=1\}}\left[\frac{5+n}{6}\frac{9+n}{3}+I_{\{\{n+3\}_2=0\}}\frac{n+2}{12}\frac{8+n}{3}+I_{\{\{n+3\}_2\neq0\}}\frac{(n+5)^2)}{36}\right]. \tag{17}
$$

Lastly, for configuration 3 part of the formula, we have

$$
\begin{aligned}
I_{\{\{n+3\}_3=2\}}(1+2+3+...+(1+\frac{n+1}{3})+\\
I_{\{\{n+3\}_2=0\}}(1+3+5+...+(1+\frac{n+1}{3}))+\\
I_{\{\{n+3\}_2\neq0\}}(2+4+6+...+(1+\frac{n+1}{3}))).
\end{aligned} \tag{18}
$$

This by applying (10) becomes

$$
I_{\{\{n+3\}_3=1\}}\left[\frac{4+n}{6}\frac{7+n}{3}+I_{\{\{n+3\}_2=0\}}\frac{(n+7)^2)}{36}+I_{\{\{n+3\}_2\neq0\}}\frac{4+n}{12}\frac{10+n}{3}\right]. \tag{19}
$$

Combining (15), (17) and (19) yields $f_3(n+3)$ defined by (2).

On the other hand, by taking arbitrary non - negative integers, the correctness of (2) and (3) can be easily verified by (1).

3 General Recursive Formula for Arbitrary N and $n >= N$

In terms of modulus operators, (1) can be redefined for different N-s. For $N = 4$, we have

$$
\begin{aligned}
f_4(n) = I_{\{\{n\}_4=0\}} \sum_{k=1}^{\frac{n}{4}+1} f_{4k-4}(3) + I_{\{\{n\}_4=1\}} \sum_{k=1}^{\frac{n-1}{4}+1} f_{4k-3}(3) + \\
I_{\{\{n\}_4=2\}} \sum_{k=1}^{\frac{n-2}{4}+1} f_{4k-2}(3) + I_{\{\{n\}_4=3\}} \sum_{k=1}^{\frac{n-3}{4}+1} f_{4k-1}(3)
\end{aligned}
\tag{20}
$$

For $N = 5$, we have

$$
\begin{aligned}
f_5(n) = I_{\{\{n\}_5=0\}} \sum_{k=1}^{\frac{n}{5}+1} f_{5k-5}(4) + I_{\{\{n\}_5=1\}} \sum_{k=1}^{\frac{n-1}{5}+1} f_{5k-4}(4) + \\
I_{\{\{n\}_5=2\}} \sum_{k=1}^{\frac{n-2}{5}+1} f_{5k-3}(4) + I_{\{\{n\}_5=3\}} \sum_{k=1}^{\frac{n-3}{5}+1} f_{5k-2}(4) + \\
I_{\{\{n\}_5=4\}} \sum_{k=1}^{\frac{n-4}{5}+1} f_{5k-1}(4)
\end{aligned}
\tag{21}
$$

For $N = 6$, we have

$$
\begin{aligned}
f_6(n) = I_{\{\{n\}_6=0\}} \sum_{k=1}^{\frac{n}{6}+1} f_{6k-6}(5) + I_{\{\{n\}_6=1\}} \sum_{k=1}^{\frac{n-1}{6}+1} f_{6k-5}(5) + \\
I_{\{\{n\}_6=2\}} \sum_{k=1}^{\frac{n-2}{6}+1} f_{6k-4}(5) + I_{\{\{n\}_6=3\}} \sum_{k=1}^{\frac{n-3}{6}+1} f_{6k-3}(5) + \\
I_{\{\{n\}_6=4\}} \sum_{k=1}^{\frac{n-4}{6}+1} f_{6k-2}(5) + I_{\{\{n\}_6=5\}} \sum_{k=1}^{\frac{n-5}{6}+1} f_{6k-1}(5)
\end{aligned}
\tag{22}
$$

In general, for an arbitrary N, we have (1)

$$f_N(n) = I_{\{\{n\}_N=0\}} \sum_{k=1}^{\frac{n}{N}+1} f_{Nk-N}(N-1) + I_{\{\{n\}_N=1\}} \sum_{k=1}^{\frac{n-1}{N}+1} f_{Nk-N+1}(N-1) + ...+ I_{\{\{n\}_N=N-1\}} \sum_{k=1}^{\frac{n-N+1}{N}+1} f_{Nk-1}(N-1) = \sum_{j=1}^{N-1} I_{\{\{n\}_N=j\}} \sum_{k=1}^{\frac{n-j}{N}+1} f_{Nk-N+j}(N-1) \tag{23}$$

References

[1] Shiryaev A.N., Problems in Probability, Springer, 2012, pp.4

[2] Shiryaev A.N., Erlikh I.G., Yaskov P.A., Probability in Theorems and Problems, pp. 12

Appendix A Scatter Configurations for $N = 3$

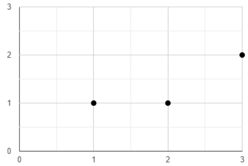

Fig. A1: $f_3(3) = 3$

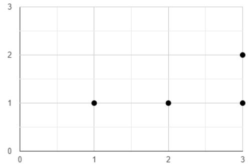

Fig. A2: $f_3(4) = 4$

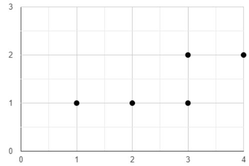

Fig. A3: $f_3(5) = 5$

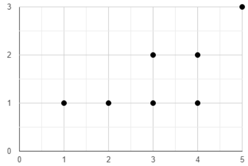

Fig. A4: $f_3(6) = 7$

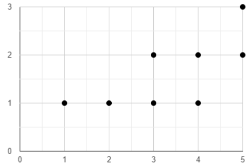

Fig. A5: $f_3(7) = 8$

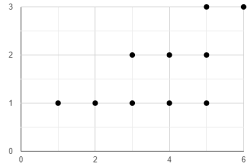

Fig. A6: $f_3(8) = 10$

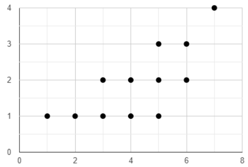

Fig. A7: $f_3(9) = 12$

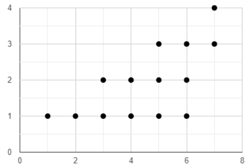

Fig. A8: $f_3(10) = 14$

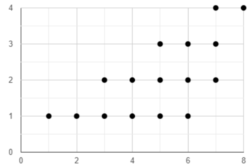

Fig. A9: $f_3(11) = 16$

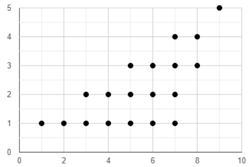

Fig. A10: $f_3(12) = 19$

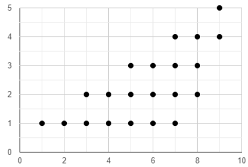

Fig. A11: $f_3(13) = 21$

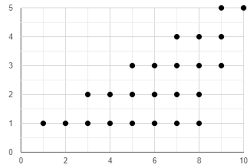

Fig. A12: $f_3(14) = 24$

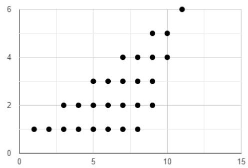

Fig. A13: $f_3(15) = 27$

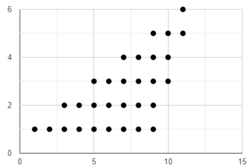

Fig. A14: $f_3(16) = 30$

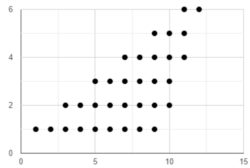

Fig. A15: $f_3(17) = 33$