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The Adomian series representation of some quadratic BSDEs

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Abstract. The representation of the solution of some Backward Stochastic Differential Equation as an infinite series is obtained. Some exactly solvable examples are considered.

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1 Introduction

In a number of papers[1,2] Adomian develops a numerical technique using special kinds of polynomials for solving non-linear functional equations. However, Adomian and his collaborators did not develop widely the problem of convergence.

In this article we will study by Adomian technique some kind of quadratic backward martingale equation and prove the convergence of the series. For example we tackle an equation of the form

$$\mathcal{E}_T(m)\mathcal{E}_T^{\alpha}(m^{\perp}) = c\exp\{\eta\} \tag{1}$$

w.r.t. stochastic integrals $m = \int f_s dW_s,\ m^{\perp} = \int g_s dW_s^{\perp}$ and real number c, where $(W, W^{\perp})$ is 2-dimension Brownian Motion and η is a random variable.

Equations of such type are arising in mathematical finance and they are used to characterize optimal martingale measures (see, Biaginiat at al (2000), Mania and Tevzadze (2000), (2003),(2006)). Note that equation (1) can be applied also to the financial market models with infinitely many assets (see M. De Donno at al (2003)). In Biagini at al (2000) an exponential equation of the form

$$\frac{\mathcal{E}_T(m)}{\mathcal{E}_T(m^\perp)} = ce^{\int_0^T \lambda_s^2 ds}$$

was considered (which corresponds to the case $\alpha = -1$).

Our goal is to show the solvability of the equation (1) using the Adomian method proving the convergence of series. On the one hand, a simpler proof of solvability is obtained. On the other hand, it allows to obtain the approximation of the solution. It is possible to find a solution in the form of series, if we define a sequence of martingales w.r.t. the measure $\mathcal{E}_T(\sum_i^n m_i + \sum_i^n m_i^\perp) \cdot P$ from equations $c'\mathcal{E}_T(m'_{n+1} + m'^{\perp}_{n+1}) = \mathcal{E}_T^2(m'^{\perp}_n)$, where $m'_{n+1} = m_{n+1} - \langle m_{n+1}, \sum_i^n, m_i \rangle$, $m'^{\perp}_{n+1} = m^\perp_{n+1} - \langle m^\perp_{n+1}, \sum_i^n m_i^\perp \rangle$, and then we write down the solution

$$m = \sum_k^\infty m_k,\ m^\perp = \sum_k^\infty m_k^\perp$$

provided the series are convergent. The proof of the convergence is greatly simplified if we present equation as a BSDE in the space of BMO-martingales and use the properties of the BMO-norm. The result is resumed in Theorem 1.

Finally we provide some examples, exactly solvable by Adomian series and also example non-solvable at all.

2 The main result

Let $(\Omega, \mathcal{F}, P)$ be a probability space with filtration $\mathbf{F} = (\mathcal{F}_t, t \in [0,T])$. We assume that all local martingales with respect to $\mathbf{F}$ are continuous. Here T is a fixed time horizon and $\mathcal{F} = \mathcal{F}_T$.

Let $\mathcal{M}$ be a stable subspace of the space of square integrable martingales H^2. Then its ordinary orthogonal $\mathcal{M}^\perp$ is a stable subspace and any element of $\mathcal{M}$ is strongly orthogonal to any element of $\mathcal{M}^\perp$ (see, e.g. [5], [6]).

We consider the following exponential equation

$$\mathcal{E}_T(m)\mathcal{E}_T^\alpha(m^\perp) = c\exp\{\eta\}, \tag{2}$$

where η is a given F_T-measurable random variable and α is a given real number. A solution of equation (2) is a triple $(c, m, m^\perp)$, where c is strictly positive constant, $m \in \mathcal{M}$ and $m^\perp \in \mathcal{M}^\perp$. Here $\mathcal{E}(X)$ is the Doleans-Dade exponential of X.

It is evident that if $\alpha = 1$ then equation (2) admits an "explicit" solution. E.g., if $\alpha = 1$ and η is bounded, then using the unique decomposition of the martingale $E(\exp\{\eta\}/F_t)$

$$E(\exp\{\eta\}/F_t) = E\exp\{\eta\} + m_t(\eta) + m_t^\perp(\eta), \quad m(\eta) \in \mathcal{M}, \quad m^\perp(\eta) \in \mathcal{M}^\perp, \tag{3}$$

it is easy to verify that the triple $c = \frac{1}{E\exp\{\eta\}}$,

$$m_t = \int_0^t \frac{1}{E(\exp\{\eta\}/F_s)} dm_s(\eta), \quad m_t^\perp = \int_0^t \frac{1}{E(\exp\{\eta\}/F_s)} dm_s^\perp(\eta)$$

satisfies equation (2).

Our aim is to prove the existence of a unique solution of equation (2) for arbitrary $\alpha \neq 0$ and η of a general structure, assuming that it satisfies the following boundedness condition:

B) η is an F_T-measurable random variable of the form

$$\eta = \bar{\eta} + \gamma A_T, \tag{4}$$

where $\bar{\eta} \in L^\infty$, γ is a constant and $A = (A_t, t \in [0,T])$ is a continuous F-adapted process of finite variation such that

$$E(var_T(A) - var_\tau(A)/F_\tau) \leq C$$

for all stopping times τ for a constant $C > 0$.

One can show that equation (2) is equivalent to the following semimartingale backward equation with the square generator

$$Y_t = Y_0 - \frac{\gamma}{2}A_t - \langle L\rangle_t - \frac{1}{\alpha}\langle L^\perp\rangle_t + L_t + L_t^\perp, \quad Y_T = \frac{1}{2}\bar{\eta}. \tag{5}$$

We use also the equivalent equation of the form

$$L_T + L_T^\perp = c + \langle L\rangle_T + \frac{1}{\alpha}\langle L^\perp\rangle_T + \frac{\gamma}{2}A_T.$$

w.r.t. $(c, L, L^\perp)$.

We use notations $|M|_{\text{BMO}} = \inf\{C : E^{\frac{1}{2}}(\langle M\rangle_T - \langle M\rangle_\tau|\mathcal{F}_\tau) \le C\}$ for BMO-norms of martingales, $|A|_\omega = \inf\{C : E(var_t^T(A)|\mathcal{F}_t) \le C\}$ for norms of finite variation processes and $A\cdot M$ for stochastic integrals.

Let us consider the system of semimartingale backward equations

$$Y_t^{(0)} = Y_0^{(0)} - \frac{\gamma}{2}A_t + L_t^{(0)} + L_t^{(0)\perp}, \quad Y_T^{(0)} = \frac{1}{2}\bar{\eta},$$

$$Y_t^{(n+1)} = Y_0^{(n+1)}$$

$$-\sum_{k=0}^{n}\langle L^{(k)}, L^{(n-k)}\rangle_t - \frac{1}{\alpha}\sum_{k=0}^{n}\langle L^{(k)\perp}, L^{(n-k)\perp}\rangle_t + L_t^{(n+1)} + L_t^{(n+1)\perp},$$

$$Y_T^{(n+1)} = 0.$$

The sequence $Y_0^{(n)} = c^{(n)}, L^{(n)} + L^{\perp(n)}, n = 0,1,2,\cdots$ can be defined consequently by the equations

$$E(\eta|\mathcal{F}_t) + \frac{\gamma}{2}E(A_T|\mathcal{F}_t) = c^{(0)} + L_t^{(0)} + L_t^{\perp(0)},$$

$$\sum_{k=0}^{n}E(\langle L^{(k)}, L^{(n-k)}\rangle_T|\mathcal{F}_t) - \frac{1}{\alpha}\sum_{k=0}^{n}E(\langle L^{(k)\perp}, L^{(n-k)\perp}\rangle_T|\mathcal{F}_t)$$

$$= c^{(n+1)} + L_t^{(n+1)} + L_t^{\perp(n+1)}.$$

Remark. If $A_t = \int_0^t a(s, W_s, B_s)ds$, then the solution of (5) is of the form $Y_t = v(t, W_t, B_t)$, where $v(t,x,y)$ is decomposed as series $\sum_n v^n(t,x,y)$ satisfying the system of PDEs

$$(\partial_t + \frac{1}{2}\Delta)v^0(t,x,y) + a(t,x,y) = 0, \quad v^0(T,x,y) = 0,$$

$$(\partial_t + \frac{1}{2}\Delta)v^n(t,x,y)$$

$$+\frac{1}{2}\sum_{k=0}^{n-1}(v_x^k(t,x,y)v_x^{n-k-1}(t,x,y) + \alpha v_y^k(t,x,y)v_y^{n-k-1}(t,x,y)) = 0,$$

$$v^n(T,x,y) = 0,\ n \ge 1.$$

Lemma 1. *Let*

$$Y_t = Y_0 + A_t + m_t, \quad Y_T = \eta,$$

where m is a martingale, $\eta \in L_\infty$ and $|A|_\omega < \infty$. Then $m \in BMO$ and

$$|m|_{\text{BMO}} \le |\eta|_\infty + |A|_\omega. \tag{6}$$

In particular, if $|A|_\omega < \infty$ *then the martingale* $E(A_T|F_t)$ *belongs to the BMO space and*

$$|E(A_T|F_\cdot)|_{\text{BMO}} \le |A|_\omega.$$

Proof. By the Ito formula

$$Y_t^2 = 2\int_0^t Y_s dm_s + 2\int_0^t Y_s dA_s + \langle m\rangle_t.$$

Taking the difference $Y_\tau^2 - Y_T^2$ and conditional expectations we have that

$$Y_\tau^2 + E(\langle m\rangle_T - \langle m\rangle_\tau|F_\tau) = E(\eta^2|F_\tau) - 2E(\int_\tau^T Y_s dA_s|F_\tau) \le$$

$$\le |\eta|_\infty^2 + 2|Y|_\infty|A|_\omega. \tag{7}$$

$E(\int_\tau^T Y_s dm_s|F_\tau) = 0$, since $Y_t \le E(\eta + |A_T - A_t||\mathcal{F}_t)$ is bounded and m is a martingale. Since the right-hand side of (7) does not depend on τ from (7) we obtain

$$|Y|_\infty^2 + ||m||_{BMO}^2 \le |\eta|_\infty^2 + |Y|_\infty^2 + |A|_\omega^2.$$

Therefore

$$||m||_{BMO}^2 \le |\eta|_\infty^2 + |A|_\omega^2,$$

which implies inequality (6).

Lemma 2. *For the BMO norms of martingales* $L^{(n)} + L^{\perp(n)}$*, defined above, the following estimates are true*

$$|L^{(n)} + L^{\perp(n)}|_{\text{BMO}} \le a_n(1+|\beta|)^n|L^{(0)} + L^{\perp(0)}|_{\text{BMO}}^{n+1}, \tag{8}$$

where the coefficients a_n *are calculating recurrently from*

$$a_0 = 1, \quad a_{n+1} = \sum_{k=0}^n a_k a_{n-k}.$$

Proof. Using Lemma 1 it is easy to show that

$$|L^{(1)} + L^{\perp(1)}|_{\text{BMO}} \le a_1(1+|\beta|)|L^{(0)} + L^{\perp(0)}|_{\text{BMO}}^2,$$

$$|L^{(2)} + L^{\perp(2)}|_{\text{BMO}} \le a_2(1+|\beta|)^2|L^{(0)} + L^{\perp(0)}|_{\text{BMO}}^3.$$

Assume that inequality (8) is valid for any $k \le n$ and let us show that

$$|L^{(n+1)} + L^{\perp(n+1)}|_{\text{BMO}} \le a_{n+1}(1+|\beta|)^{n+1}|L^{(0)} + L^{\perp(0)}|^{n+2}_{\text{BMO}}. \tag{9}$$

Applying Lemma 1 for $Y_t^{(n+1)}$ and the Kunita-Watanabe inequality we have

$$|L^{(n+1)} + L^{\perp(n+1)}|_{\text{BMO}} \le$$

$$\le \operatorname*{ess\,sup}_{\tau} \sum_{k=0}^{n} E(var_\tau^T(\sum_{k}^{n}\langle L^{(k)}, L^{(n-k)}\rangle + \beta\langle L^{\perp(k)}, L^{\perp(n-k)}\rangle)|\mathcal{F}_\tau)$$

$$\le \sum_{k=0}^{n} \operatorname*{ess\,sup}_{\tau} E^{\frac{1}{2}}(var_\tau^T\langle L^{(k)}\rangle|\mathcal{F}_\tau)E^{\frac{1}{2}}(var_\tau^T\langle L^{\perp(n-k)}\rangle|\mathcal{F}_\tau)$$

$$+|\beta|\sum_{k=0}^{n} \operatorname*{ess\,sup}_{\tau} E^{\frac{1}{2}}(var_\tau^T\langle L^{\perp(k)}\rangle|\mathcal{F}_\tau)E^{\frac{1}{2}}(var_\tau^T\langle L^{\perp(n-k)}\rangle|\mathcal{F}_\tau)$$

$$\le \sum_{k}^{n} |L^{(k)}|_{\text{BMO}}|L^{(n-k)}|_{\text{BMO}} + |\beta||L^{\perp(k)}|_{\text{BMO}}|L^{\perp(n-k)}|_{\text{BMO}}$$

$$\le (1+|\beta|)\sum_{k=0}^{n}|L^{(k)} + L^{\perp(k)}|_{\text{BMO}}|L^{(n-k)} + L^{\perp(n-k)}|_{\text{BMO}}. \tag{10}$$

Therefore, from (10), using inequalities (8) for any $k \le n$, we obtain

$$|L^{(n+1)} + L^{\perp(n+1)}|_{\text{BMO}} \le$$

$$\le (1+|\beta|)\sum_{k=0}^{n} a_k(1+|\beta|)^k|L^{(0)}+L^{\perp(0)}|^{k+1}_{\text{BMO}}a_{n-k}(1+|\beta|)^{n-k}||L^{(n-k)}+L^{\perp(n-k)}|^{n-k+1}_{\text{BMO}}$$

$$\le (1+|\beta|)^{n+1}|L^{(0)} + L^{\perp(0)}|^{n+2}_{\text{BMO}}\sum_{k=0}^{n} a_k a_{n-k} =$$

$$= a_{n+1}(1+|\beta|)^{n+1}|L^{(0)} + L^{\perp(0)}|^{n+2}_{\text{BMO}}$$

and the validity of inequality (8) follows by induction.

Theorem 1. *The series* $\sum_{n\ge0}(L^{(n)} + L^{\perp(n)})$ *is convergent in BMO-space, if* γ *and* $|\bar\eta|_\infty$ *are small enough and the sum of series is a solution of the equation (5).*

Proof. Without loss of generality assume that $\eta = 0$. Using the lemma 2 we get

$$|L^{(n)} + L^{\perp(n)}|_{\mathrm{BMO}} \le a_n(1+|\beta|)^n |L^{(0)} + L^{\perp(0)}|^{n+1}_{\mathrm{BMO}} \le a_n(1+|\beta|)^n |\gamma A|^{n+1}_{\omega}.$$

By lemma 3 of appendix, since

$$\overline{\lim}_{n\to\infty} \sqrt[n]{a_n} = \overline{\lim}_{n\to\infty} \sqrt[n]{\frac{1}{2n+1} C^{2n+2}_{n+1}} = \overline{\lim}_{n\to\infty} \sqrt[n]{\frac{(2n)!}{n!n!}} = \overline{\lim}_{n\to\infty} \sqrt[n]{\frac{(2n)^{2n}}{n^{2n}}} = 4,$$

the series is convergent, when $\gamma < \frac{1}{4|A|_\omega(1+|\beta|)}$.

Remark. Since $\max(|L|_{\mathrm{BMO}}, |L^\perp|_{\mathrm{BMO}}) \le |L+L^\perp|_{\mathrm{BMO}} \le |L|_{\mathrm{BMO}} + |L^\perp|_{\mathrm{BMO}}$ the convergence $\sum_{n\ge0}(L^{(n)} + L^{\perp(n)})$ implies convergence of $\sum_{n\ge0} L^{(n)}$ and $\sum_{n\ge0} L^{\perp(n)}$ and vice versa.

The existence of the solution for arbitrary bounded η is proven [8]. We can prove here little more general result

Proposition 1. *There exists solution of (2) for sufficiently small* γ *and arbitrary bounded* $\bar\eta$ *.*

Proof. Let $\bar m + \bar m^\perp$ be solution of (2) for $\eta = \gamma A_T$ and sufficiently small γ. From the result of [8] there exists a solution of

$$\mathcal{E}_T(\tilde m)\mathcal{E}^\alpha_T(\tilde m^\perp) = c\exp\{\bar\eta\},$$

w.r.t

$$\bar P = \mathcal{E}_T(\bar m + \bar m^\perp)\cdot,\ \tilde m + \tilde m^\perp \in \mathcal{M}(F,\bar P) + \mathcal{M}^\perp(F,\bar P).P.$$

It is easy to verify that $m + m^\perp = \bar m + \bar m^\perp + \tilde m + \tilde m^\perp$ is a solution of (2) for $\eta = \bar\eta + \gamma A_T$.

The uniqueness of the solution was proved in [8].

Proposition 2. . *Let* η *be an* $\mathcal{F}_T$ *-measurable random variable. If there exists a triple* $(c, m, m^\perp)$*, where* $c \in R_+, m \in BMO\cap\mathcal{M}, m^\perp \in BMO\cap\mathcal{M}^\perp$ *satisfying equation (2) then such solution is unique.*

We now show that without finiteness of $|A|_\omega$ either the solution does not exists or the convergence of series is valid in a week sense.

Example 1. Let $\alpha = -1,\ \gamma = 2,\ \bar\eta = 0,\ A_t = \frac12\int_0^t (W_s^2 + W_s^{2\perp})ds,\ \mathbf{F} = (\mathcal{F}_t^{W,W^\perp})$, where $W, W^\perp$ is 2-dimensional Brownian motion. Then (5) becomes

$$L_T + L_T^\perp = c + \langle L\rangle_T - \langle L^\perp\rangle_T + \frac12\int_0^T (W_s^2 + W_s^{2\perp})ds.$$

We have

$$L_T^{(0)} + L_T^{(0)\perp} = c_0 + \int_0^T (T-s) W_s dW_s + \int_0^T (T-s) W_s^\perp dW_s^\perp,$$

$$L_T^{n+1)} + L_T^{(n+1)\perp} = c_n + \sum_{k=0}^n \langle L^{(k)}, L^{(n-k)} \rangle_T - \sum_{k=0}^n \langle L^{(k)\perp}, L^{(n-k)\perp} \rangle_T, \; n \geq 0.$$

Let assume

$$L_T^{(n)} = \int_0^T (T-s)^{2n+1} \alpha_n W_t dW_s,$$
$$L_T^{(n)\perp} = \int_0^T (T-s)^{2n+1} \beta_n W_t^\perp dW_s^\perp.$$

Then $a_0 = 1, \;\; \beta_0 = 1$ and

$$L_T^{(n+1)} = c_n' + \sum_{k=0}^n \int_0^T (T-s)^{2n+2} \alpha_k \alpha_{n-k} W_s^2 ds$$
$$L_T^{(n+1)\perp} = c_n'' - \sum_{k=0}^n \int_0^T (T-s)^{2n+2} \beta_k \beta_{n-k} W_s^2 ds, \; n \geq 0.$$

Taking stochastic derivatives $D_t, D_t^\perp$ and conditional expectations on both sides we get

$$(T-s)^{2n+3} \alpha_n W_t = 2 \sum_{k=0}^n \alpha_k \alpha_{n-k} W_t \int_t^T (T-s)^{2n+2} ds$$
$$= \frac{2}{2n+3} W_t (T-t)^{2n+3} \sum_{k=0}^n \alpha_k \alpha_{n-k},$$
$$(T-s)^{2n+3} \beta_n W_t^\perp = -\frac{2}{2n+3} W_t^\perp (T-t)^{2n+3} \sum_{k=0}^n \beta_k \beta_{n-k},$$

which means that

$$\alpha_{n+1} = \frac{2}{2n+3} \sum_{k=0}^n \alpha_k \alpha_{n-k}, \; \beta_{n+1} = -\frac{2}{2n+3} \sum_{k=0}^n \beta_k \beta_{n-k}, n \geq 0.$$

Introducing $\alpha(s) = \sum_{n=0}^{\infty} \alpha_n s^{2n+1},\ \beta(s) = \sum_{n=0}^{\infty} \beta_n s^{2n+1}$ one obtains

$$\alpha'(s) = \alpha_0 + \sum_{n=0}^{\infty} (2n+3)\alpha_{n+1} s^{2n+2}$$

$$= 1 + 2\sum_{n=0}^{\infty}\sum_{k=0}^{n} (\alpha_k \alpha_{n-k}) s^{2n+2} = 1 + 2a^2(s),$$

$$\beta'(s) = \beta_0 + \sum_{n=0}^{\infty} (2n+3)\beta_{n+1} s^{2n+2}$$

$$= 1 - 2\sum_{n=0}^{\infty}\sum_{k=0}^{n} \beta_k \beta_{n-k} s^{2n+2} = 1 - 2\beta^2(s).$$

I.e.

$$\begin{aligned} \alpha'(s) &= 1 + 2a^2(s),\ \alpha(0) = 0, \\ \beta'(s) &= 1 - 2\beta^2(s),\ \beta(0) = 0. \end{aligned} \tag{11}$$

Thus

$$\alpha(s) = \frac{1}{\sqrt{2}} \tan(\sqrt{2}s),\ \beta(s) = -\frac{1}{\sqrt{2}} \tanh(\sqrt{2}s).$$

If $T < \frac{\pi}{2\sqrt{2}}$ series are convergent (not in BMO-space) and $(c, L, L^{\perp})$ is defined as $c = \frac{1}{2} \ln \cos(\sqrt{2}T)\cosh(\sqrt{2}T)$ (by calculations in the appendix),

$$L_t = \frac{1}{\sqrt{2}} \int_0^t \tan(\sqrt{2}s) W_s dW_s,\ L_t^{\perp} = -\frac{1}{\sqrt{2}} \int_0^t \tanh(\sqrt{2}s) W_s^{\perp} W_s^{\perp}.$$

When $T > \frac{\pi}{2\sqrt{2}}$ a local martingale L satisfying $L_T - \langle L \rangle_T = \frac{1}{2} \int_0^T W_t^2 dt$ does not exist (despite the fact that $\int_0^T W_t^2 dt$ is p-integrable for each $p \geq 1$), since from $\mathcal{E}_T(2L) = e^{\int_0^T W_t^2 dt}$ follows that $E e^{\int_0^T W_t^2 dt} = E\mathcal{E}_T(2L) \leq 1$, which contradicts to $E e^{\int_0^T W_t^2 dt} = \infty$ (see appendix).

In the next example exact solution of (5) also exists, however it does not belong to the extreme cases considered in [9],[10].

Example 2. Let $\alpha = -1,\ \gamma = 2,\ \bar{\eta} = 0,\ A_t = \int_0^t W_s W_s^{\perp} ds,\ \mathbf{F} = (\mathcal{F}_t^{W,W^{\perp}})$, where $W, W^{\perp}$ is a 2-dimensional Brownian motion. Then (5) becomes

$$L_T + L_T^{\perp} = c + \langle L \rangle_T - \langle L^{\perp} \rangle_T + \int_0^T W_s W_s^{\perp} ds.$$

We have

$$L_T^{(0)} = EL_T^{(0)} + \int_0^T (T-s)W_s^\perp dW_s,\ L_T^{(0),\perp} = EL_T^{(0),\perp} + \int_0^T (T-s)W_s dW_s^\perp,$$

$$L_T^{(n+1)} + L_T^{(n+1)\perp} = c_n + \sum_{k=0}^n \langle L^{(k)}, L^{(n-k)}\rangle_T - \sum_{k=0}^n \langle L^{(k)\perp}, L^{(n-k)\perp}\rangle_T,\ n \geq 0.$$

We assert that

$$L_T^{(n)} = EL_T^{(n)} + \int_0^T (T-s)^{2n+1}(\alpha_n W_t + \beta_n W_s^\perp)dW_s,$$
$$L_T^{(n)\perp} = EL_T^{(n)\perp} + \int_0^T (T-s)^{2n+1}(\beta_n W_t - \alpha_n W_s^\perp)dW_s^\perp,$$

where $\alpha_0 = 0,\ \ \beta_0 = 1$ and

$$\alpha_{n+1} = \frac{2}{2n+3}\sum_{k=0}^n(\alpha_k\alpha_{n-k} - \beta_k\beta_{n-k}),\ \beta_{n+1} = \frac{4}{2n+3}\sum_{k=0}^n \alpha_k\beta_{n-k}, n \geq 0.$$

Indeed,

$$L_T^{(n+1)} + L_T^{(n+1)\perp} = c_n$$
$$+\sum_{k=0}^n \int_0^T (T-s)^{2n+2}(\alpha_k W_s + \beta_k W_s^\perp)(\alpha_{n-k}W_s + \beta_{n-k}W_s^\perp)ds$$
$$-\sum_{k=0}^n \int_0^T (T-s)^{2n+2}(\beta_k W_s - \alpha_k W_s^\perp)(\beta_{n-k}W_s - \alpha_{n-k}W_s^\perp)ds$$
$$= \sum_{k=0}^n \int_0^T (T-s)^{2n+2}[(\alpha_k\alpha_{n-k} - \beta_k\beta_{n-k})W_s^2 - (\alpha_k\alpha_{n-k} - \beta_k\beta_{n-k})W_s^{\perp 2}$$
$$+2(\alpha_k\beta_{n-k} + \beta_k\alpha_{n-k})W_s W_s^\perp]ds + c_n,\ n \geq 0.$$

Using representation of integrands by stochastic derivatives we get

$$
\begin{aligned}
&(T-t)^{2n+3}(\alpha_{n+1}W_t+\beta_{n+1}W_t^{\perp})\\
&= E[D_t(\sum_{k=0}^{n}\langle L^{(k)},L^{(n-k)}\rangle_T-\sum_{k=0}^{n}\langle L^{(k)\perp},L^{(n-k)\perp}\rangle_T)|\mathcal{F}_t]\\
&= 2\sum_{k=0}^{n}[(\alpha_k\alpha_{n-k}-\beta_k\beta_{n-k})W_t+(\alpha_k\beta_{n-k}+\beta_k\alpha_{n-k})W_t^{\perp}]\int_t^T(T-s)^{2n+2}ds\\
&= \frac{2(T-t)^{2n+3}}{2n+3}\sum_{k=0}^{n}[(\alpha_k\alpha_{n-k}-\beta_k\beta_{n-k})W_t+(\alpha_k\beta_{n-k}+\beta_k\alpha_{n-k})W_t^{\perp}],\\
&(T-t)^{2n+3}(\beta_{n+1}W_t-\alpha_{n+1}W_t^{\perp})\\
&= E[D_t^{\perp}(\sum_{k=0}^{n}\langle L^{(k)},L^{(n-k)}\rangle_T-\sum_{k=0}^{n}\langle L^{(k)\perp},L^{(n-k)\perp}\rangle_T)|\mathcal{F}_t]\\
&= 2\sum_{k=0}^{n}[-(\alpha_k\alpha_{n-k}-\beta_k\beta_{n-k})W_t^{\perp}+(\alpha_k\beta_{n-k}+\beta_k\alpha_{n-k})W_t]\int_t^T(T-s)^{2n+2}ds\\
&= \frac{2(T-t)^{2n+3}}{2n+3}\sum_{k=0}^{n}[-(\alpha_k\alpha_{n-k}-\beta_k\beta_{n-k})W_t^{\perp}+(\alpha_k\beta_{n-k}+\beta_k\alpha_{n-k})W_t].
\end{aligned}
$$

Equalising coefficients at $W, W^{\perp}$ we obtain the desired formula. One can be checked that $\lim_{n\to\infty}\sqrt[n]{|a_n|}=0, \lim_{n\to\infty}\sqrt[n]{|b_n|}=0$. Introducing $\alpha(s)=\sum_{n=0}^{\infty}\alpha_n s^{2n+1},\ \beta(s)=\sum_{n=0}^{\infty}\beta_n s^{2n+1}$ one obtains

$$
\begin{aligned}
L_t &= L_0+\int_0^t(\alpha(T-s)W_s+\beta(T-s)W_s^{\perp})dW_s,\\
L_t^{\perp} &= L_0^{\perp}+\int_0^t(\beta(T-s)W_s-\alpha(T-s)W_s^{\perp})dW_s^{\perp}.
\end{aligned}
$$

On the other hand we can derive ODE for the pair (α,β)

$$
\begin{aligned}
\alpha'(s) &= 2\alpha^2(s)-2\beta^2(s),\ \alpha(0)=0, \qquad (12)\\
\beta'(s) &= 1+4\alpha(s)\beta(s),\ \beta(0)=0.
\end{aligned}
$$

Indeed

$$\alpha'(s) = \alpha_0 + \sum_{n=0}^{\infty}(2n+3)\alpha_{n+1}s^{2n+2}$$

$$= 2\sum_{n=0}^{\infty}\sum_{k=0}^{n}(\alpha_k\alpha_{n-k} - \beta_k\beta_{n-k})s^{2n+2} = 2a^2(s) - 2\beta^2(s),$$

$$\beta'(s) = \beta_0 + \sum_{n=0}^{\infty}(2n+3)\beta_{n+1}s^{2n+2}$$

$$= 1 + 4\sum_{n=0}^{\infty}\sum_{k=0}^{n}\alpha_k\beta_{n-k}s^{2n+2} = 1 + 4\alpha(s)\beta(s).$$

The equation (12) is easy to solve, if we pass to the equation for complex-variable function $\zeta(s) = \alpha(s) + i\beta(s)$

$$\zeta'(s) = i + 2\zeta^2(s),\ \zeta(0) = 0.$$

It is obvious that $\zeta(s) = \frac{1}{1-i}\tan((1+i)s)$ is a solution. We have

$$\zeta(s) = \frac{1}{2}(1+i)\frac{\sin((1+i)s)\cos((1-i)s)}{|\cos((1+i)s)|^2}$$

$$= \frac{1}{4}(1+i)\frac{\sin(2s) + i\sinh(2s)}{|\cos((1+i)s)|^2}$$

$$= \frac{1}{4}\frac{\sin(2s) - \sinh(2s) + i(\sin(2s) + \sinh(2s))}{\cos^2(s)\cosh^2(s) + \sin^2(s)\sinh^2(s)}.$$

Finally we can write explicit solution

$$\alpha(s) = \frac{1}{4}\frac{\sin(2s) - \sinh(2s)}{\cos^2(s)\cosh^2(s) + \sin^2(s)\sinh^2(s)},$$

$$\beta(s) = \frac{1}{4}\frac{\sin(2s) + \sinh(2s)}{\cos^2(s)\cosh^2(s) + \sin^2(s)\sinh^2(s)}$$

of (12) and conclude that it exists on whole $[0,\infty)$, since the denominator does not vanish.

A Appendix

The formula $Ee^{-T^2\int_0^1 W_t^2 dt} = \frac{1}{\sqrt{\cosh(\sqrt{2}T)}}$ is derived in [7]. Similarly we can prove

Proposition 3.

$$Ee^{\int_0^T W_t^2dt} = \begin{cases} \frac{1}{\sqrt{\cos(\sqrt{2}T)}}, \text{ if } T < \frac{\pi}{2\sqrt{2}} \\ \infty, \text{ if } T \geq \frac{\pi}{2\sqrt{2}} \end{cases}.$$

Proof. Let $e_n(t)$ be orhonormal basis in $L^2[0,1]$. Then $Ee^{\int_0^T W_t^2dt} = Ee^{T^2\int_0^1 W_t^2dt} = Ee^{T^2\sum_{n=1}^\infty(\int_0^1 e_n(t)W_tdt)^2} = E\prod_{n=1}^\infty e^{T^2(\int_0^1 e_n(t)W_tdt)^2}$. Since

$$E(\int_0^1 e_n(t)W_tdt)(\int_0^1 e_m(t)W_tdt) = \int_0^T e_n(t)\int_0^T (t\wedge s)e_m(s)dsdt$$

it is convenient to use the orthonormal basis of eigenvectors of the operator $\int_0^T (t\wedge s)f(s)ds$ in $L^2[0,1]$. From $\lambda f(t) = \int_0^T (t\wedge s)f(s)ds$ follows that $\lambda f''(t) = -f(t),\ f(0) = 0,\ f'(1) = 0$. The function $\sin\mu\pi t$ satisfies these conditions iff $\mu^2 = 1/\lambda$, $\cos\mu\pi = 0$ and $\mu = -1/2+n$. Thus

$$\lambda_n = \frac{1}{(n-1/2)^2\pi^2},\ e_n(t) = \sqrt{2}\sin((n-1/2)\pi t),\ n\geq 1$$

and $E(\int_0^1 e_n(t)W_tdt)(\int_0^1 e_m(t)W_tdt) = \lambda_n\int_0^1 e_n(t)e_m(t)dt = 0,\ n\neq m$. Since random variables $(\int_0^1 e_n(t)W_tdt)$ are orthogonal and normal they are also independent. Hence taking into account infinite product decomposition of $\cos(\sqrt{2}t)$ one gets

$$Ee^{\int_0^T W_t^2dt} = \prod_{n=1}^\infty Ee^{T^2(\int_0^1 e_n(t)W_tdt)^2}$$
$$= \prod_{n=1}^\infty Ee^{T^2\lambda_n W_1^2} = \prod_{n=1}^\infty \frac{1}{\sqrt{1-\frac{2T^2}{(n-1/2)^2\pi^2}}}$$
$$= \sqrt{\prod_{n=1}^\infty \frac{1}{1-\frac{8T^2}{(2n-1)^2\pi^2}}} = \frac{1}{\sqrt{\cos(\sqrt{2}T)}},$$

if $\sqrt{2}T < \pi/2$.
It easy to see that

$$E\exp\left(\int_0^{\frac{\pi}{2\sqrt{2}}} W_t^2dt\right) = \lim_{T\uparrow\frac{\pi}{2\sqrt{2}}} E\exp\left(\int_0^T W_t^2dt\right) = \lim_{T\uparrow\frac{\pi}{2\sqrt{2}}}\frac{1}{\sqrt{\cos(\sqrt{2}T)}} = \infty.$$

If $T > \frac{\pi}{2\sqrt{2}}$ then $Ee^{\int_0^T W_t^2 dt} > Ee^{\int_0^{\frac{\pi}{2\sqrt{2}}} W_t^2 dt} = \infty$.

Lemma 3. *Let $(a_n)_{n\geq 0}$ be a solution of the system*

$$a_0 = 1,\ a_{n+1} = \sum_{k=0}^{n} a_k a_{n-k}. \tag{13}$$

Then $a_n = \frac{1}{4n+2}\binom{2n+2}{n+1}$.

Proof. For the series $u(\lambda) = \sum_{n=0}^{\infty} a_n \lambda^n$ from (13) we get equation $u(\lambda) = 1 + \lambda u^2(\lambda)$, with the roots $u(\lambda) = \frac{1}{2\lambda}(1 \pm \sqrt{1-4\lambda})$. The equality $u(\lambda) = \frac{1}{2\lambda}(1 + \sqrt{1-4\lambda})$ is impossible, since decomposition of the right hand side is starting from the term $\frac{1}{\lambda}$. Therefore, equality $a_n = \frac{1}{4n+2}\binom{2n+2}{n+1}$ follows from the Taylor expansion of $1 - \sqrt{1-4\lambda}$, since

$$u(\lambda) = \frac{1}{2\lambda}(1 - \sqrt{1-4\lambda})$$

$$= -\frac{1}{2}\sum_{n\geq 1} \frac{\frac{1}{2}(\frac{1}{2}-1)\cdots(\frac{1}{2}-n+1)}{n!}(-4)^n \lambda^{n-1}$$

$$= \frac{1}{2}\sum_{n\geq 1} \frac{(2-1)\cdots(2n-2-1)}{2^n n!} 4^n \lambda^{n-1}$$

$$= \frac{1}{2}\sum_{n\geq 1} \frac{(2n-3)!!}{n!} 2^n \lambda^{n-1} = \frac{1}{2}\sum_{n\geq 1} \frac{1}{2n-1}\binom{2n}{n}\lambda^{n-1}.$$

Lemma 4. *There exist sequences $(m_i, i \geq 1) \in \mathcal{M}$, $(m_i^{\perp}, i \geq 1) \in \mathcal{M}^{\perp}$, such that $e^{\eta} = c_1 \frac{\mathcal{E}_T(m_1)}{\mathcal{E}_T(m_1^{\perp})}\mathcal{E}_T^2(m_1^{\perp})$ and*

$$e^{\eta} = c_n \frac{\mathcal{E}_T(\sum_i^n m_i)}{\mathcal{E}_T(\sum_i^n m_i^{\perp})}\mathcal{E}_T^2(m_n'^{\perp}),\ n \geq 2, \tag{14}$$

where $m_n'^{\perp} = m_n^{\perp} - \langle m_n^{\perp}, \sum_i^{n-1} m_i^{\perp}\rangle$.

Proof. The theorem will be proved by induction. Assume (14) is valid for n. There exist such martingales $m_{n+1}, m_{n+1}^{\perp}$ that $c'\mathcal{E}_T(m'_{n+1} + m'^{\perp}_{n+1}) = \mathcal{E}_T^2(m_n'^{\perp})$ and

$$m'_{n+1} = m_{n+1} - \langle m_{n+1}, \sum_i^n, m_i\rangle,\ \ m'^{\perp}_{n+1} = m^{\perp}_{n+1} - \langle m^{\perp}_{n+1}, \sum_i^n m_i^{\perp}\rangle$$

are martingales w.r.t. $\mathcal{E}(\sum_i^n m_i + m_i^\perp)\cdot P$. Thus

$$e^\eta = c_n c' \frac{\mathcal{E}_T(\sum_i^n m_i)}{\mathcal{E}_T(\sum_i^n m_i^\perp)}\mathcal{E}(m_{n+1} - \langle m_{n+1}, \sum_i^n m_i\rangle + m_{n+1}^\perp - \langle m_{n+1}^\perp, \sum_i^n m_i^\perp\rangle)$$

$$= c_{n+1}\frac{\mathcal{E}_T(\sum_i^n m_i)\mathcal{E}_T(m_{n+1} - \langle m_{n+1}, \sum_i^n m_i\rangle)}{\mathcal{E}_T(\sum_i^n m_i^\perp)\mathcal{E}_T(m_{n+1}^\perp - \langle m_{n+1}^\perp, \sum_i^n m_i^\perp\rangle)}\mathcal{E}_T^2(m_{n+1}^\perp - \langle m_{n+1}^\perp, \sum_i^n m_i^\perp\rangle)$$

$$= c_{n+1}\frac{\mathcal{E}_T(\sum_i^{n+1} m_i)}{\mathcal{E}_T(\sum_i^{n+1} m_i^\perp)}\mathcal{E}_T^2(m'^\perp_{n+1}).$$

Remark. If we will prove the convergence of series $\sum_i m_i, \sum_i m_i^\perp$, then $m_n^\perp \to 0, m'^\perp_n \to 0,\ \mathcal{E}(m'^\perp_n) \to 1$ and $e^\eta = c\frac{\mathcal{E}_T(\sum_i^\infty m_i)}{\mathcal{E}_T(\sum_i^\infty m_i^\perp)}$.

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