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# How to Compute the Gradient of the Analytically Unknown Value Function

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## Section 1. The Basic Idea of the Research Project

It is well known that vast majority of the real-world optimization problems cannot be solved analytically in closed form since they are highly nonlinear by their intrinsic nature.

Denote $V(x)$ the real−valued **Value Function** of the optimization problem, where we minimize corresponding **Objective Function** $F(x, z)$ over certain parametric set $Z$ and the argument $x$ belongs to some domain $D$ of the $n$-dimensional Euclidean space $R^n$.

Numerical methods suggest to approximate analytically unknown **Value Function** $V(x)$ on a dense discrete subset $G$ of grid points by the function $V(x, h)$, where $x$ belongs to the discrete grid set $G$ and the parameter $h$ shows the "denseness" of $G$ with respect to the domain $D$ and is defined as the smallest positive number satisfying the following condition: for arbitrary point $y$ belonging to domain $D$ one can find a point $x$ belonging to the grid set $G$ such that $|y - x| < h$.

Many real-world problems also require the approximate computation of the **Gradient** $grad\, V(x)$, that is the vector of all partial derivatives of the function $V(x)$. In general it turns out to be very difficult problem to construct sophisticated algorithm to approximate $grad\, V(x)$, as the corresponding difference quotients start wild oscillations when the parameter $h$ tends to $0$ and one finds out soon that the latter quotients converge nowhere in the limit.

**Our basic observation:** The **Value function** $V(x)$ of the optimization problem is often convex (or semi convex) in multidimensional argument $x$ (for example, in engineering thermodynamics it is **the Convex Envelope of the Gibbs free energy function**). Therefore we should use the advantage of **Convexity** to construct convergent numerical approximations to $grad\ V(x)$.

**Our basic idea:** Assume that $V(x)$ is a convex function. Replace the approximation $V(x,h)$ by some convex approximation $C(x,h)$ in a hope that the latter one **will better imitate** the shape of the unknown convex function $V(x)$ and hence the **gradient** *grad* $C(x,h)$ can be announced as the reasonable approximation to the unknown $grad\ V(x)$!

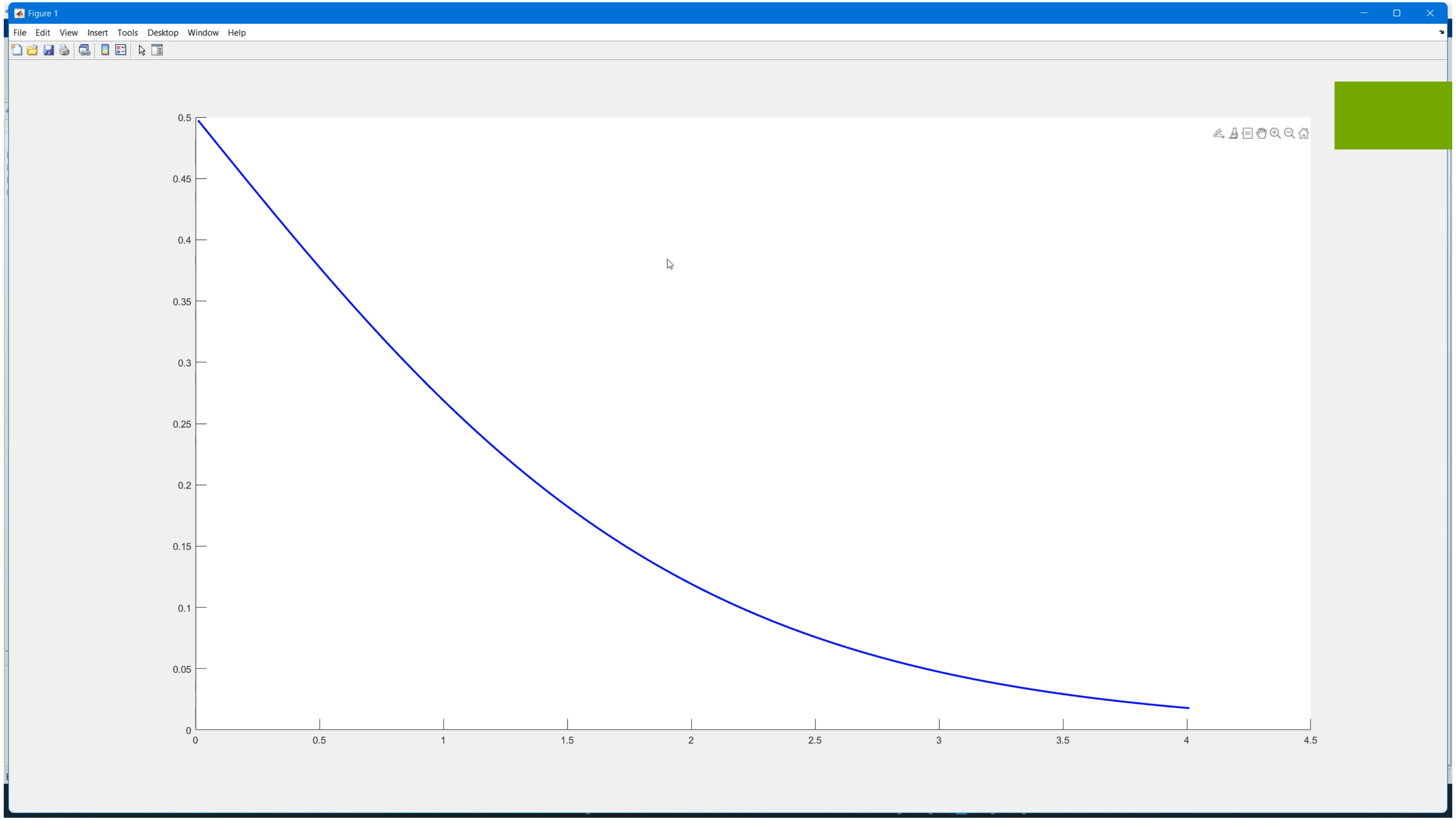
The clever choice of convex approximation consists in constructing the so called **Discrete Convex Envelope** denoted by $D\ conv\ V(x,h)$ of the function $V(x,h)$, which is defined on a domain $D$ as the maximal convex function dominated by the function $V(x,h)$ **on a discrete set of grid points** $G$. The construction of **the discrete convex envelope** is carried out by several algorithms in computational geometry and most popular among them is **QHULL** (the quick hull algorithm for convex hulls), which finds the **convex hull** of arbitrary finite set of points in $n$- dimensional Euclidean space $R^n$ and the **discrete convex envelope is obtained as a "lower part" of the corresponding convex hull** !

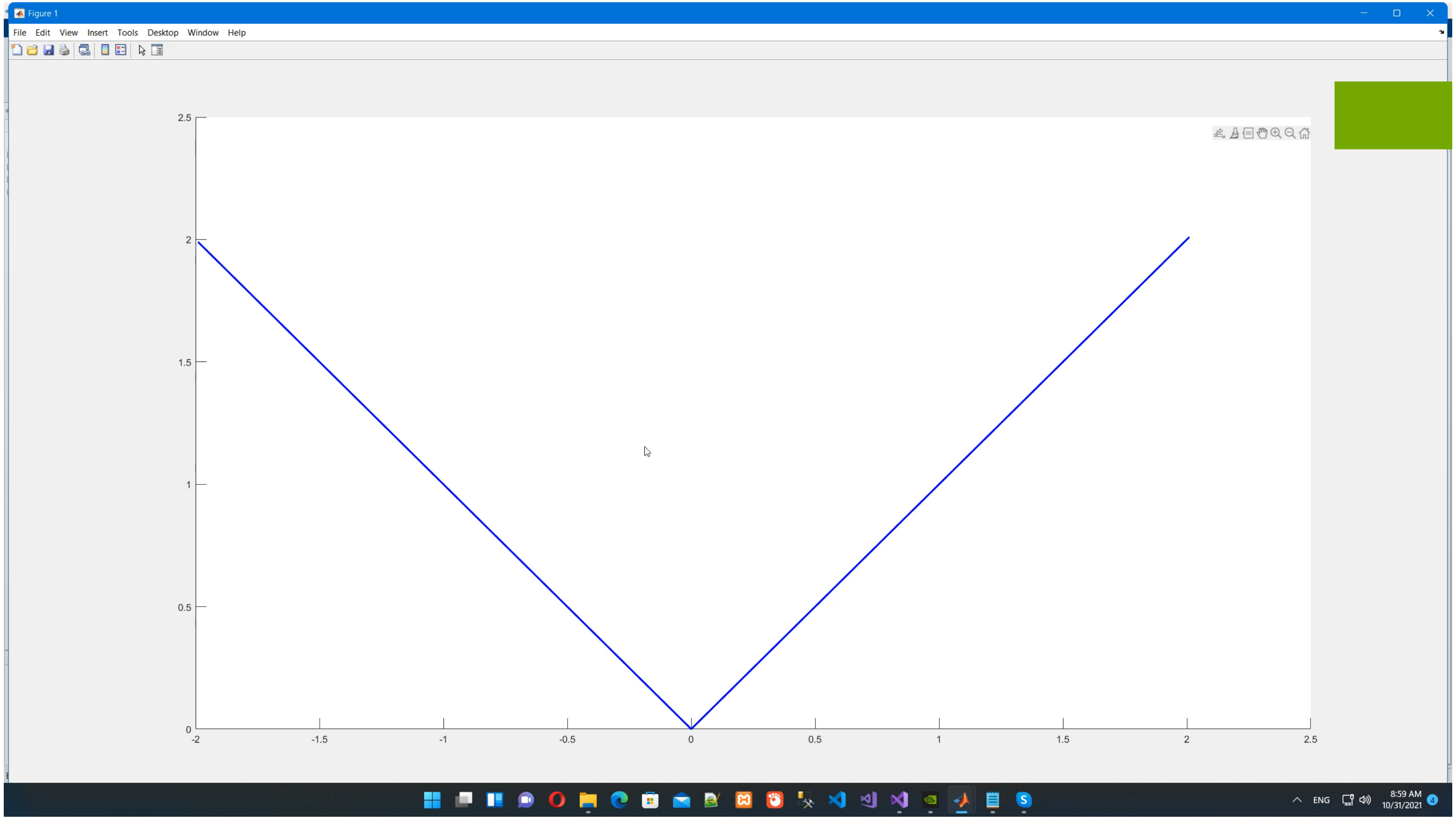
**Our basic idea** seems intuitively reasonable, but it needs rigorous mathematical justification. The latter justification has been given in our published paper

**Shashiashvili K., Shashiashvili M.** From the uniform approximation of a solution of the PDE to the $L^2$-approximation of the gradient of the solution. *J. Convex Anal.* **21** (2014), no. 1, 237-252,

where we have given rigorous mathematical justification of our intuitive arguments proving new type reverse Poincare inequalities for the **difference of two semi convex functions as well as for the difference of two convex envelopes of arbitrary continuous objective functions** not assuming even existence of first order partial derivatives of the **latter functions, see Proposition 3.2 and Theorem 3.3 therein**.

# Section 2. Convex Envelope Animations

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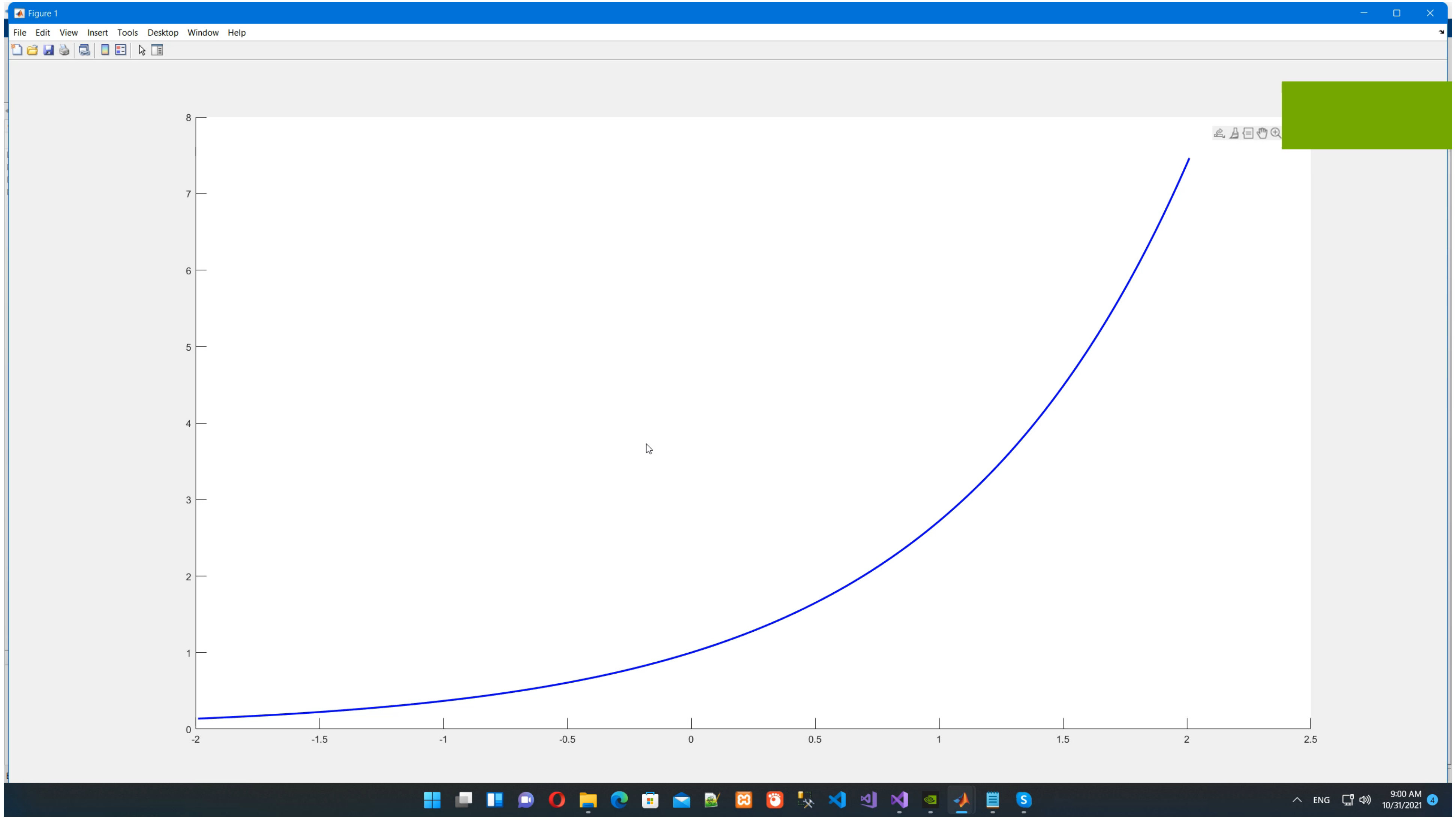

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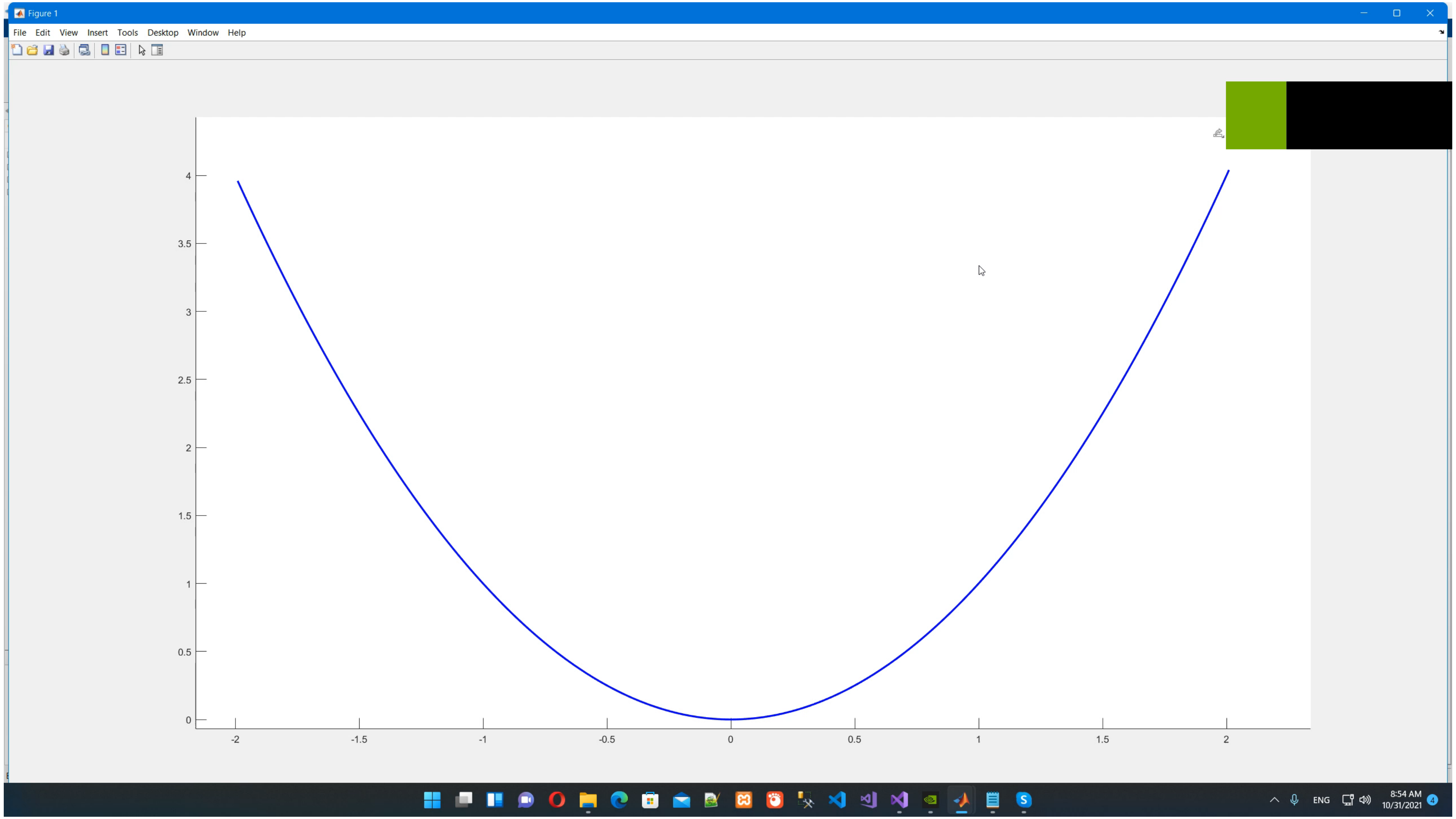
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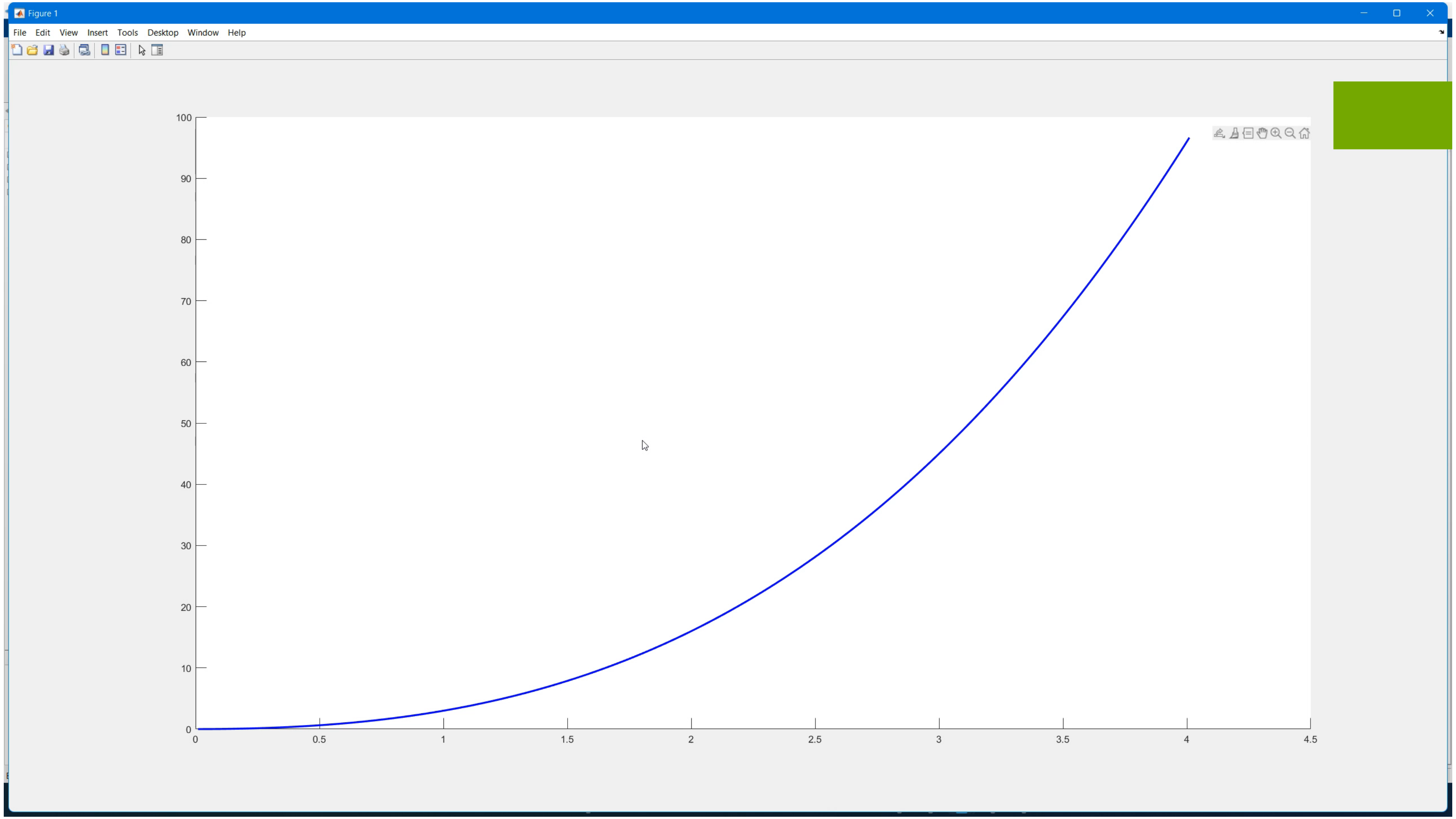
Figure 1
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## Section 3. The $L^2$-Approximation of the Gradient of the Semiconvex Function through the Convex Envelope

Let $u: D \to R$ be analytically unknown viscosity solution of the nonlinear second order elliptic partial differential equation

$$F(x, u, grad\ u, Hess\ u) = 0 \qquad (3.1)$$

in a bounded open convex subset $D$ of $R^n$.

As pointed out in the introduction the solution of the equation (3.1) turns out to be semiconvex (or semiconcave) function if the latter equation is related to different kind of optimization problems.

Suppose the bounded viscosity solution $u$ of equation (3.1) is semiconvex function and we are given its uniform continuous numerical approximation $u_\delta: D \to R$, where $\delta$ is a small parameter, which typically measures the mesh size. The objective consists in constructing interior $L^2$-approximation of the unknown Sobolev gradient $grad\ u$ based on the uniform approximation $u_\delta$. Moreover, it is desirable to estimate the gradient's $L^2$-error through the $L^\infty(D)$-uniform error of approximation.

We shall see in this section that such a construction is possible and it uses two ingredients: the energy inequality (2.5) in Shashiashvili M. and Shashiashvili K. [9] and the notion of the convex envelope.

The convex envelope $conv(u)$ of a bounded continuous function $u$ in $D$ is defined as the supremum of all convex functions which are majorized by the function $u$

$$conv(u) = \sup\{v(x): v(x)\ convex\ in\ D, v(x) \le u(x)\ for\ all\ x \in D\}\,. \tag{3.2}$$

The mapping $u \to conv(u)$ possesses some nice properties which we prove below

**Lemma 3.1.** *The mapping $u \to conv(u)$ has Lipschitz property*

$$\|conv(u) - conv(v)\|_{L^\infty(D)} \le \|u - v\|_{L^\infty(D)} \tag{3.3}$$

*If only $u$, $v$ belong to $C(D) \cap L^\infty(D)$.*

**Proof.** Denote

$$d = \|u - v\|_{L^\infty(D)},$$

Then we have $-d \le u(x) - v(x) \le d$, i.e. $v(x) - d \le u(x)$, $u(x) - d \le v(x)$.

Hence we have

$$conv(v) - d \le u, conv(u) - d \le v.$$

This means that the convex functions $conv(v) - d$ and $conv(\mathrm{u}) - d$ are majorized respectively by $u$, $v$ .

By the definition of the convex envelope we obtain

$$conv(v) - d \le conv(u), \qquad conv(u) - d \le conv(v),$$

i.e. $-d \le conv\ u(x) - conv\ v(x) \le d$, thus we derive the inequality (3.3). □

Taking successively $u = 0$ and $v = 0$ in (3.3) we get

$$\begin{cases} \|conv(v)\|_{L^\infty(D)} \le \|v\|_{L^\infty(D)}, \\ \|conv(u)\|_{L^\infty(D)} \le \|u\|_{L^\infty(D)}. \end{cases} \tag{3.4}$$

**Proposition 3.2.** *On the space $C(D) \cap L^\infty(D)$ the mapping $u \to conv(u)$ possesses the following important property*

$$\int_D |grad\ conv(u) - grad\ conv(v)|^2 \cdot \frac{d_{\partial D}^2}{n} dx \le 5meas\ D \cdot \|u - v\|_{L^\infty(D)} \big(\|u\|_{L^\infty(D)} + \|v\|_{L^\infty(D)}\big). \tag{3.5}$$

**Proof.** We have from the bound (3.4) that the convex functions $conv(u)$ and $conv(v)$ are bounded, thus we can apply the energy inequality (2.5) for the latter convex functions and get

$$\int_D |grad\ conv(u) - grad\ conv(v)|^2 \cdot \frac{d_{\partial D}^2}{n} dx$$
$$\le 5meas\ D \cdot \|conv(u) - conv(v)\|_{L^\infty(D)} \big(\|conv(u)\|_{L^\infty(D)} + \|conv(v)\|_{L^\infty(D)}\big). \tag{3.6}$$

The assertion follows after application of Lemma 3.1 and the bound (3.4). □

Consider now the bounded viscosity solution $u$ of the equation (3.1) which is assumed to be semiconvex with semiconvexity constant $c \geq 0$ and its uniform continuous numerical approximation $u_\delta$, i.e.

$$\|u_\delta - u\|_{L^\infty(D)} \xrightarrow[\delta \to 0]{} 0. \tag{3.7}$$

Further consider the bounded continuous functions

$$u + c \cdot v_0 \quad \text{and} \quad u_\delta + c \cdot v_0 \tag{3.8}$$

and their convex envelopes

$$\operatorname{conv}(u + c \cdot v_0) \quad \text{and} \quad conv(u_\delta + c \cdot v_0), \tag{3.9}$$

where

$$v_0(x) = \frac{1}{2} \cdot |x|^2.$$

The next proposition is the main result of Section 3.

**Theorem 3.3.** *The following weighted $L^2$-estimate is valid for the unknown $grad\, u$ through the function $grad(conv(u_\delta + c \cdot v_0) - c \cdot v_0)$*

$$\int_D |grad(conv(u_\delta + c \cdot v_0) - c \cdot v_0) - grad\, u|^2 \cdot \frac{d_{\partial D}^2}{n} dx$$
$$\leq 5 meas\, D \|u_\delta - u\|_{L^\infty(D)} \big(2\|u\|_{L^\infty(D)} + \|u_\delta - u\|_{L^\infty(D)} + 2c\|v_0\|_{L^\infty(D)}\big). \quad (3.10)$$

**Proof.** Let us apply Proposition 3.2 to the functions $u_\delta + c \cdot v_0$ and $u + c \cdot v_0$ , we shall have

$$\int_D |grad\, conv(u_\delta + c \cdot v_0) - grad\, conv(u + c \cdot v_0)|^2 \cdot \frac{d_{\partial D}^2}{n} dx$$
$$\leq 5 meas\, D \|u_\delta - u\|_{L^\infty(D)} \big(\|u_\delta + c \cdot v_0\|_{L^\infty(D)} + \|u + c \cdot v_0\|_{L^\infty(D)}\big). \quad (3.11)$$

By the semiconvexity criteria (2.3) in Shashiashvili M. and Shashiashvili K. [9] we have that the function $u + c \cdot v_0$ is convex and therefore coincides with its convex envelope $conv(u + c \cdot v_0)$, hence we get

$$grad\, conv(u + c \cdot v_0) = grad(u + c \cdot v_0) = grad\, u + grad(c \cdot v_0),$$

the rest is obvious. □

Thus the $L^2$-approximation problem of the unknown $grad\, u$ is reduced to the efficient numerical computation of convex envelope $conv(u_\delta + c \cdot v_0)$ and its gradient. We note here that if the solution of PDE (3.1) is convex the unknown $grad\, u$ is approximated by the $grad\, conv(u_\delta)$.

## Section 4. Computation of the Gradient of the Solution of Monge-Ampere Partial Differential Equation in a Planar Domain

We discuss next the Monge-Ampere equation. The Monge-Ampere equation is a fully nonlinear elliptic PDE. Applications of the Monge-Ampere equation appear in the classical problem of prescribed Gauss curvature and in the problem of optimal mass transportation (with quadratic cost).

We shall present a simple (nine point stencil) finite difference method which performs well for smooth as well as for singular solutions. The Monge-Ampere PDE in a planar domain $D \subset R^2$ is the following

$$det\big(Hessian\ U(x)\big) = f(x), f(x) \geq 0,$$

or equivalently

$$\frac{\partial^2 u}{\partial x^2} \cdot \frac{\partial^2 u}{\partial y^2} - \left(\frac{\partial^2 u}{\partial x \partial y}\right)^2 = f \quad \text{with Dirichlet boundary conditions } u = g \text{ on } \partial D \tag{4.1}$$

and the additional convexity constraint

$$u(x, y) \quad \text{is convex in } D, \tag{4.2}$$

which is required for the equation to be elliptic. Without the convexity constraint this equation does not have a unique solution. For example, taking the boundary function $g = 0$, if $u$ is a solution, then $-u$ is also a solution.

The numerical method involves simply discretizing the second derivatives using standard central differences on a uniform Cartesian grid. The result is

$$\left(D_{xx}^2 u_{ij}\right) \cdot \left(D_{yy}^2 u_{ij}\right) - \left(D_{xy}^2 u_{ij}\right)^2 = f_{ij}, \tag{4.3}$$

where

$$\begin{cases} D_{xx}^2 u_{ij} = \dfrac{u_{i+1,j} + u_{i-1,j} - 2u_{ij}}{h^2}, \\ D_{yy}^2 u_{ij} = \dfrac{u_{i,j+1} + u_{i,j-1} - 2u_{ij}}{h^2}, \\ D_{xy}^2 u_{ij} = \dfrac{u_{i+1,j+1} + u_{i,j-1} - u_{i-1,j+1} - u_{i-1,j-1}}{4h^2}. \end{cases} \tag{4.4}$$

Introduce the notation

$$a_1 = \frac{u_{i+1,j} + u_{i-1,j}}{2}, \qquad a_2 = \frac{u_{i,j+1} + u_{i,j-1}}{2}, \qquad a_3 = \frac{u_{i+1,j+1} + u_{i,j-1}}{2}, \qquad a_4 = \frac{u_{i-1,j+1} + u_{i-1,j-1}}{2} \tag{4.5}$$

and rewrite (4.3) as a quadratic equation for $u_{ij}$:

$$4\left(a_1 - u_{ij}\right)\left(a_2 - u_{ij}\right) - \frac{1}{4}(a_3 - a_4)^2 = h^4 f_{ij}. \tag{4.6}$$

Now solving for $u_{ij}$ and selecting the smaller one (in order to select the locally convex solution), we obtain

$$u_{ij} = \frac{1}{2}(a_1 + a_2) - \frac{1}{2}\sqrt{(a_1 - a_2)^2 + \frac{1}{4}(a_3 - a_4)^2 + h^4 f_{ij}}. \tag{4.7}$$

We can now use Gauss-Seidel iteration to find the fixed point of (4.7).

The Dirichlet boundary conditions are enforced at boundary grid points. The convexity constraint (4.2) is not enforced (beyond the selection of the positive root in (4.7).

Next we consider two exact solutions for the Monge-Ampere PDE (4.1), (4.2) on the square $[0,1] \times [0,1]$.

**Example 4.1.**

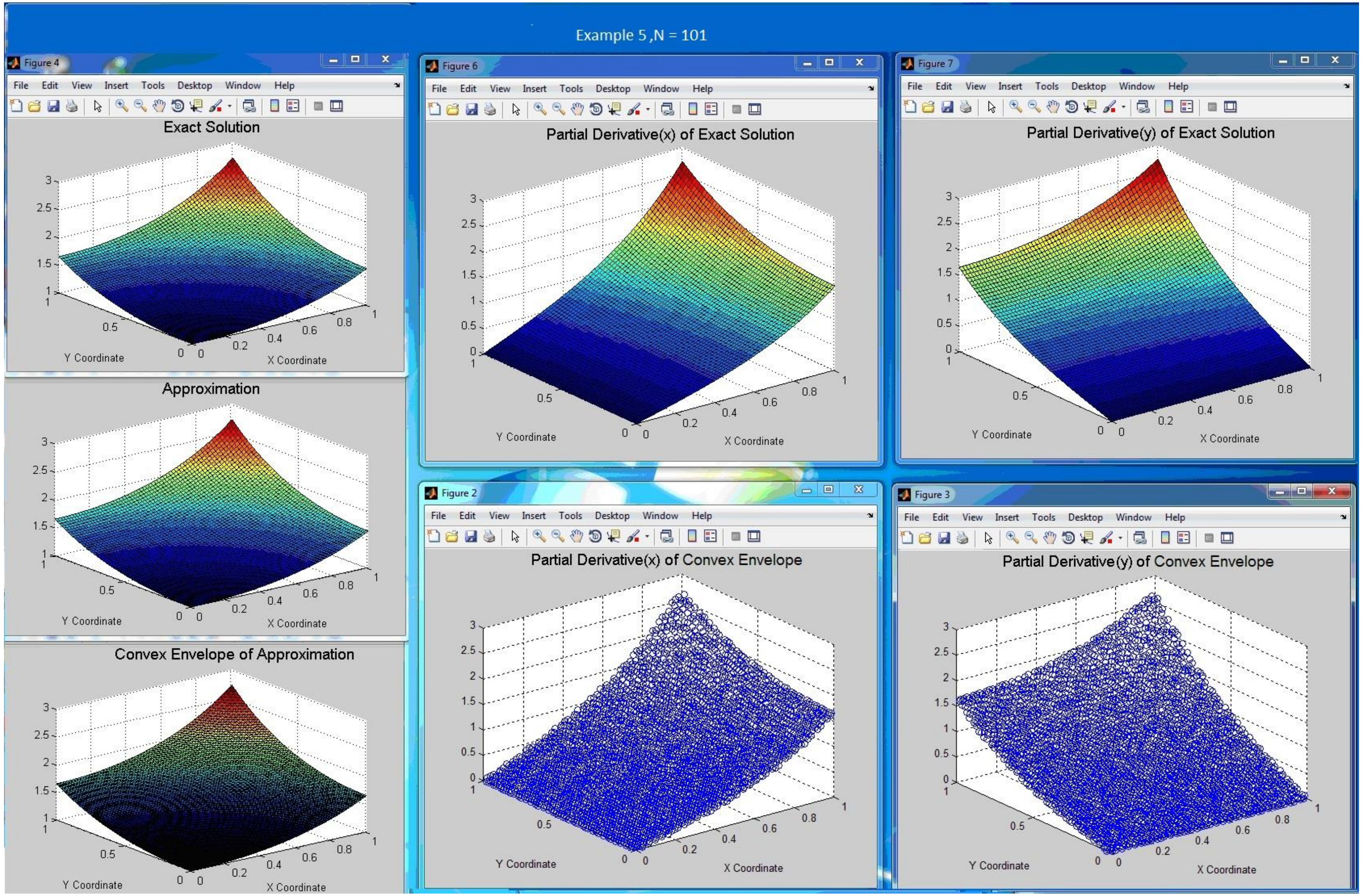
$$\begin{cases} u(x,y) = \exp\left(\dfrac{x^2+y^2}{2}\right), \\ f(x,y) = (1+x^2+y^2)\cdot\exp(x^2+y^2). \end{cases}$$

**Example 4.2.**

$$\begin{cases} u(x,y) = \dfrac{2\sqrt{2}}{3}(x^2+y^2)^{3/4}, \\ f(x,y) = \dfrac{1}{\sqrt{x^2+y^2}}. \end{cases}$$

In this example the function $f$ blows up at the boundary point $(0,0)$.

We note that we use fast algorithm to accelerate computations in the finite difference method (4.3)-(4.7).



Example 5 ,N = 101
Figure 4
Exact Solution
Approximation
Convex Envelope of Approximation
Figure 6
Partial Derivative(x) of Exact Solution
Figure 7
Partial Derivative(y) of Exact Solution
Figure 2
Partial Derivative(x) of Convex Envelope
Figure 3
Partial Derivative(y) of Convex Envelope
Y Coordinate
X Coordinate


**Figure 1**

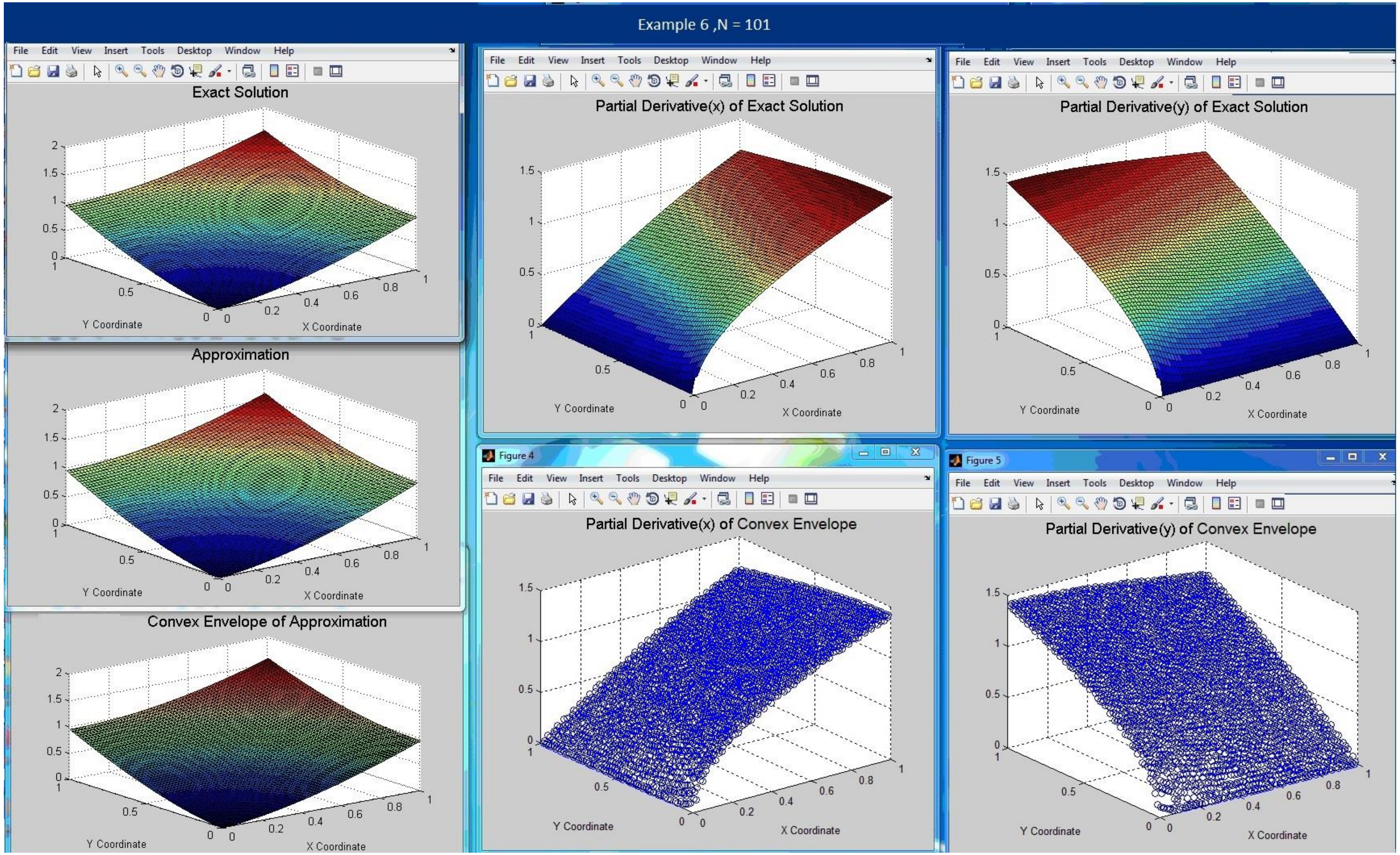



**Figure 2**

The Monge-Ampere equations (the Examples 4.1 and 4.2) are considered on the square $[0,1] \times [0,1]$.

In the tables below for the different grid points we compute the number of iterations, the computation times, the errors of approximation of the exact solution and of the exact gradient.

Computation times and errors for the exact solution and its gradient for the Example 4.1 on an $N \times N$ grid:

| # | Number of iterations | Computation times | Uniform error for the exact solution | Uniform error for the exact gradient | $L^2$-error for the exact gradient |
|---|---|---|---|---|---|
| 21 | 1362 | 1 sec. | $1.5 \times 10^{-4}$ | 0.1255 | 0.011 |
| 61 | 10840 | 10 sec. | $1.8 \times 10^{-5}$ | 0.0441 | 0.0038 |
| 101 | 28764 | 60 sec. | $6.7 \times 10^{-6}$ | 0.0267 | 0.0023 |
| 141 | 54802 | 300 sec. | $3.4 \times 10^{-6}$ | 0.0192 | 0.0016 |

Computation times and errors for the exact solution and its gradient for the Example 4.2 on an  grid:

| # | Number of iterations | Computation times | Uniform error for the exact solution | Uniform error for the exact gradient | $L^2$-error for the exact gradient |
|---|---|---|---|---|---|
| 21 | 1397 | 1 sec. | $1.5 \times 10^{-4}$ | 0.1511 | 0.0077 |
| 61 | 11065 | 10 sec. | $1 \times 10^{-4}$ | 0.0887 | 0.0027 |
| 101 | 29312 | 70 sec. | $4.9 \times 10^{-5}$ | 0.0689 | 0.0016 |
| 141 | 55768 | 300 sec. | $2.9 \times 10^{-5}$ | 0.0583 | 0.0011 |

We give the surface plots (for Examples 4.1 and 4.2) of the following functions:

a) the exact solution,

b) finite difference numerical approximation,

c) the convex envelope of the numerical approximation,

d) partial derivative w.r. to $x$ of the exact solution,

e) partial derivative w.r. to $y$ of the exact solution,

f) partial derivative w.r. to $x$ of the convex envelope,

g) Partial derivative w.r. to $y$ of the convex envelope.

## Section 5. Pricing and Hedging of American Options written on Multiple Assets

In this section we study the multidimensional parabolic obstacle problem and its relation to the pricing and hedging of American options written on multiple assets. We shall consider the so called strong solutions of parabolic obstacle problem that have been studied, for example, in Friedman [3, Chapter 1]. Strong solutions have second order Sobolev (weak) derivatives so that the Partial Differential Equation (PDE) can be written pointwisely a.e., strong solutions should be preferable in financial applications because of their better regularity properties.

The above obstacle problem appears naturally in the valuation of American type Claims in financial market. The obstacle is the so called payoff function and the solution of the obstacle problem is the value function of the American option written on multiple assets. A good background study is given in the paper by Broadie and Detemple [1].

The semiconvexity is a natural property of a large class of value functions of the optimization problems (see, for instance, Cannarsa and Sinestrari [2]).

This convexity (semiconvexity) of the value function of the American option for arbitrary fixed time instant is the starting point of our new method of the construction of the nearly optimal discrete time delta hedging strategies for American options.

American option can be exercised by its holder (as an opposite to European option) at any time up to and including expiry. This makes their pricing mathematically challenging and few closed form solutions have been found. American options are important because they are very widely traded. At least as important as the pricing of American options are the hedging issues that are crucial for the writer of the option.

In this section we study the parabolic obstacle problem in the strong sense. More precisely, we seek a solution $u(x,t)$, which belongs to the parabolic Sobolev space (see, for example, Krylov [6, Chapter 2]) and satisfies a system of inequalities

$$\begin{cases} Lu(x,t) \le 0, \qquad u(x,t) \ge g(x), \\ Lu(x,t) \cdot \big(u(x,t) - g(x)\big) = 0 \end{cases} \tag{5.1}$$

$(dx \times dt)$ with terminal condition

$$u(x,T) = g(x), \tag{5.2}$$

where $g(x)$, $x \in R^n$ is a given non-negative continuous function representing an obstacle and $Lu$ is the second order linear parabolic differential operator

$$Lu(x,t) = \sum_{i,j=1}^{n} a_{ij}(x,t) \cdot \frac{\partial^2 u(x,t)}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x,t) \cdot \frac{\partial u(x,t)}{\partial x_i} - r(t) \cdot u(x,t) + \frac{\partial u(x,t)}{\partial t}, \tag{5.3}$$

when the obstacle $g(x)$ is non-smooth there are not many known techniques to be used in the study of the obstacle problem. Our objective is to develop some new results for the nonsmooth case, with focus on applications to American type options written on multiple assets, which is an active research area at present in mathematical finance.

We will consider the pricing and hedging of multidimensional American options in a financial market driven by a general multidimensional Ito diffusion. The American option is a financial contract, assuming a time horizon of $T > 0$ and a market consisting of $n$ assets $S(t) = \big(S_1(t), \dots, S_n(t)\big)$ giving a payoff at time $t$ equal to $\Psi\big(S_1(t), \dots, S_n(t)\big)$ where $\Psi(x)$ is a non-negative continuous function from $R_+^n$ to $R_+$ defining the contract. The American option corresponding to this claim gives the owner of the option the right (but not the obligation) to exercise the option at any time $\tau$, $0 \leq \tau \leq T$. At the exercise time $\tau$, the owner of the option receives an amount equal to $\Psi\big(S(\tau)\big)$. We suppose the existence of a positive and continuous instantaneous interest rate $r(t)$ and also of the dividend rates $d_i(t)$ of the assets $S_i(t)$, $i = 1, \dots, n$.

We assume that there exists a risk-neutral martingale measure $Q$, such that with respect to $Q$ the logarithms of the prices $X(t) = \left(\ln\left(S_1(t)\right), \dots, \ln\left(S_n(t)\right)\right)$ solve a system of stochastic differential equations

$$dX(t) = b(X(t), t) \cdot dt + \sigma(X(t), t) \cdot dW(t), \qquad X(0) = x, \qquad 0 \le t \le T, \tag{5.4}$$

where

$$b_i(x, t) = r(t) - d_i(t) - \frac{1}{2}\sum_{k=1}^{n} \sigma_{ik}^2, \qquad i = 1, \dots, n, \tag{5.5}$$

Here $W(t) = \left(W_1(t), \dots, W_n(t)\right)$ is a standard $n$- dimensional Brownian motion with respect to the filtration $(\mathfrak{I}_t)_{0 \le t \le T}$ defined on a probability space $(\Omega, \mathfrak{I}, Q)$, $\sigma(x, t) = \left(\sigma_{ij}(x, t)\right)_{i,j=1,\dots,n}$, where

$$a_{ij}(x, t) = \frac{1}{2}\sum_{k=1}^{n} \sigma_{ik}(x, t) \cdot \sigma_{jk}(x, t). \tag{5.6}$$

We will assume that the operator $Lu$ is uniformly parabolic in the sense that there exists $\lambda > 0$ such that

$$\sum_{i,j=1}^{n} a_{ij}(x,t) \cdot \xi_i \cdot \xi_j \geq \lambda \cdot |\xi|^2, \qquad \text{whenever} \quad (x,t) \in R^n \times [0,T] \quad \text{and} \quad \xi \in R^n. \tag{5.7}$$

We will assume also that the functions $b(x,t)$ and $\sigma(x,t)$ are bounded and Lipschitz continuous, that is, there exists a constant $c > 0$ such that for all $(x, \tilde{x}) \in R^n$ and $(s,t) \in [0,T]$ we have

$$\|\sigma(x,t) - \sigma(\tilde{x},s)\| + \|b(x,t) - b(\tilde{x},s)\| \leq c \cdot (\|x - \tilde{x}\| + |t-s|). \tag{5.8}$$

We will impose the basic assumption on the payoff function:

$$\Psi(x), \qquad x \in R_+^n \quad \text{is a nonnegative Lipschitz continuous convex function.} \tag{5.9}$$

Denote $V(x,t)$, $x \in R_+^n$, $0 \leq t \leq T$, the value function of the American option at time $t$, if the underlying assets are trading at $\big(S_1(t), \dots, S_n(t)\big) = (x_1, \dots, x_n)$. Then it is well known that

$$u(x,t) = V(\exp(x), t), \qquad x \in R^n, \qquad 0 \leq t \leq T, \tag{5.10}$$

is a unique solution of the parabolic obstacle problem (5.1), (5.2) with the obstacle function

$$g(x) = \Psi\big(\exp(x)\big), \qquad x \in R^n, \tag{5.11}$$

The convexity (semiconvexity) of the value function $V(x,t)$ of the American option for arbitrary fixed time instant $t$ is the crucial point for our new device of the construction of the nearly optimal discrete time delta hedging strategies for American options written on multiple assets.

Indeed recently in the paper by Shashiashvili M. and Shashiashvili K. [9], we have developed a novel devise of numerical computation of the gradient of the analytically unknown function provided that the latter function is convex (or semiconvex) and we have already constructed its some uniform approximation. It is based on a new weighted inequality in Mathematical Analysis found by us (called otherwise the reverse Poincare inequality) for the difference of two semiconvex functions.

In this project we investigate the discrete time hedging problem for the American option written on the multiple underlying assets $S(t) = \big(S_1(t), \dots, S_n(t)\big)$, $0 \le t \le T$ and having a nonnegative convex payoff function $\Psi(x)$, $x \in R_+^n$.

It is a classical mathematical result at present (see, for example, Karatzas and Shreve [5, Chapter 2]) that for the perfect hedging in continuous time the writer of the option should construct the so called delta-hedging portfolio, which means that at an arbitrary time instant the should hold $grad\, V(S(t),t)$ units of the underlying assets, where $V(x,t)$ denotes the value function of the American option and $grad\, V(x,t)$ is a vector of its partial derivatives with respect to the components of its multidimensional space argument $x, x \in R_+^n$.

But the perfect hedging in continuous time requires the continuous rebalancing of the writer's portfolio in the underlying assets and the money market account, which is impossible in practice. In reality, the writer trades only at some discrete instants of time at which he rebalances his portfolio. Moreover, the delta-hedging requires the knowledge of the gradient $grad\,V(x,t)$ of the value function $V(x,t)$, but the explicit form neither of the value function, nor of its partial derivatives is known even in the simplest Black-Sholes model for American put option with finite horizon $T > 0$.

Several approximation methods were devised in order to compute the value function of the American option. In particular, finite difference methods were developed in Wilmott, Dewynne, and Howison [10], and Jaillet, Lamberton, and Lapeyre [4]. We assume here that we have already been given some continuous in the argument $x$ uniform approximation $V_h(x,t)$ to the unknown value function $V(x,t)$ of the American option at the equidistant rebalancing times $t_k = k \cdot \delta$, $\delta = \frac{T}{N}$, $k = 0,1,\dots,N$ (for example, the Bermudan option value function approximation), where $h$ is a certain small parameter indicating the error of approximation. In particular, we assume that the following bound is valid uniformly in $k$, $k = 0,1,\dots,N$,

$$\sup_{x\in R_+^n}|V_h(x,t_k) - V(x,t_k)| \le c \cdot h, k = 0,1,\dots,N, \tag{5.12}$$

Here $c$ is some positive constant depending on the parameters of our model and the payoff function $\Psi(x)$ and we naturally assume that

$$V_h(x,T) = \Psi(x), \qquad x \in R_+^n. \tag{5.13}$$

Our hedging method consists in the following: for each function $V_h(x, t_k)$, $k = 1,2, \dots, N$, consider first its convex envelope $conv\ V_h(x, t_k)$, $k = 1,2, \dots, N$, which is the maximal convex function dominated by the given function $V_h(x, t_k)$ and then its gradient $grad\ conv\ V_h(x, t_k)$, $k = 1,2, \dots, N$. Now the discrete time hedge $D_{\delta,h}(t)$, $0 \le t \le T$ can be defined in the following manner

$$D_{\delta,h}(t) = grad\ conv\ V_h(S(t_k), t_k) \ \text{ if } \ t_k \le t < t_{k+1}, \qquad k = 1, \dots, (N-1). \tag{5.14}$$

Our basic idea is to use the latter discrete time hedge as a reasonable approximation to the unknown continuous time optimal delta-hedge

$$D(t) = grad\ V(S(t), t), \qquad 0 \le t \le T. \tag{5.15}$$

Denote $\Pi_{\delta,h}(t)$, $0 \le t \le T$ the value process of the discrete time hedging portfolio and $\Pi(t)$, $0 \le t \le T$, respectively, the value process of continuous time optimal delta-hedging portfolio. Then the error due to our discrete time hedge is equal to

$$E^Q \sup_{0 \le t \le T} \left| \Pi_{\delta,h}(t) - \Pi(t) \right|. \tag{5.16}$$

One of the objectives of this research project consists in estimating the latter error for American options written on multiple assets and proving that it converges to zero when discretization parameters $\delta$ and $h$ tend to zero. We should note here that this program has been successfully carried out in one dimensional case for Black-Sholes model in Shashiashvili and Hussain [7]. The estimation of the error (5.16) for multi asset American option problem will heavily rely on the weighted reverse Poincare inequalities in $R^n$ and $R^n_+$ and therefore proving such kind of inequalities is one of the objectives of this research project. We formulate the latter inequality in $R^n$.

Let $U(x)$ and $V(x)$ be two semiconvex functions in $R^n$ with the semiconvexity constants $c_U$ and $c_V$, respectively (see Cannarsa and Sinestrari [2, Chapter 1, Definition 1.1.1]) and $H(x)$ be a nonnegative twice continuously differentiable weight function. Then the following weighted reverse Poincare inequality should be valid (under certain conditions on $U(x)$, $V(x)$ and $H(x)$ ) for the difference $U(x) - V(x)$

$$\int_{R^n} |grad\, U(x) - grad\, V(x)|^2 \cdot H(x) dx$$

$$\leq c\|U - V\|_{L^\infty(R^n)} \int_{R^n} |U(x) + V(x) + max(c_U, c_V) \cdot |x|^2| \cdot |\Delta H(x)| dx\,, \tag{5.17}$$

where Δ denotes the Laplace operator and $c$ is the absolute constant.

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**Thank you very much for your attention!**