A Fish Growth Model with Random Extreme Weight

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Abstract. The generalized von Bertalanffy growth model with random extreme weight is expressed as a unique solution of a Backward Stochastic Differential Equation.

Keywords: Bertalanffy growth model, Brownian Motion, Subordinator, Backward Equation.

1 Introduction

We use a Backward Stochastic Differential Equation (BSDE) approach to generalize the deterministic von Bertalanffy fish growth model [9], which is widely used as a descriptive model for size-at-age data.

The von Bertalanffy model, formulated for the case of decreasing growth with age, is a differential equation with a linearly decreasing growth rate, or equivalently,

$$\frac{dL}{dt} = K(L_\infty - L_t), \quad L_\infty > 0 \quad K > 0, \tag{1}$$

with initial condition $L(t_0) = 0$. The solution to differential equation (1) is

$$L_t = L_\infty[1 - e^{-\alpha(t-t_0)}], \tag{2}$$

where L_∞ is the asymptotic upper bound of the variable under study, which is only attained as time tends to infinity; α is the curvature parameter, or von Bertalanffy growth rate, which determines the speed at which the fish approaches L_∞ and t_0 - denotes the (theoretical) time at which the fish has zero length and can take negative values.

A general expression for the von Bertalanffy curve, also called "generalized von Bertalanffy growth curve" (see [1], [5], [7] and references therein), is

$$W_t = W_\infty[1 - e^{-K(t-t_0)}]^p, \tag{3}$$

where the parameter p can be known or unknown. For example, the value $p = 1$ corresponds to the von Bertalanffy growth curve (2) and is mainly used for modeling the length variable whereas a generalized version, including a new parameter $p \geq 1$, allows for modeling both length and weight for some animal species in both isometric ($p = 3$) and allometric ($p \neq 3$) situations. However, although the deterministic von Bertalanffy model may reasonably represent the mean growth pattern of a population, it ignores both environmental shocks and genetic heterogeneity, which are fundamental components of real biological processes. To account for stochasticity, several authors have introduced random perturbations or stochastic growth increments.

Several stochastic length and weight growth models are available in the literature. Some individual-based stochastic models of growth (see, e.g., [2], [3]) are proposed using stochastic differential equations of the type

$$L_t = L_0 + \int_0^t a(s, L_s)ds + \int_0^t \sigma(s, L_s)dW_s,$$

where L_t is the size at time t, $a(t, L_t)$ characterizes the deterministic intrinsic growth of the individual, $\sigma(t, L_t)$ gives the magnitude of the random fluctuations and W_t is a Brownian Motion.

In [2] and [3], individual-based stochastic models of growth are proposed using stochastic differential equations. In particular, Lv and Pitchford [3] consider three different stochastic models for fish length with a constant asymptotic length, two of which are linear. In all these models, the mean growth of length follows the von Bertalanffy growth function. For example, in the first model proposed by [3], the fish length is described by the stochastic differential equation

$$dL_t = \alpha(L_\infty - L_t)dt + \sigma(L_\infty - L_t)dB_t \tag{4}$$

whose unique solution is

$$L_t = L_\infty(1 - \mathcal{E}_t\left(-\alpha s + \sigma B\right)), \tag{5}$$

where α, σ, L_∞ are constant, B is a Brownian motion and $\mathcal{E}_t(X)$ denotes the stochastic exponential of the semimartingale X.

In this formulation, stochasticity acts multiplicatively on the growth rate, introducing random fluctuations around the deterministic von Bertalanffy trajectory while preserving its mean behavior. Specifically, the drift term ensures that the expected length EL_t still satisfies the deterministic von Bertalanffy equation, whereas the diffusion term, scaled by σ, is intended to account both for the environment and the inter-individual variability.

Russo et al. [8] proposed a growth model for fish (and other animals) in which growth is modeled as the solution of a linear stochastic differential equation driven by a Lévy process with positive jumps (a subordinator). The unique solution of this equation is the stochastic exponential of the Lévy process. The model exhibits several desirable features and is the first stochastic growth model with increasing paths, thus yielding a more realistic stochastic description of individual growth.

The model in Russo et al. [8] is given by the process Y_t, which is the solves the stochastic differential equation (SDE)

$$dY_t = (L_\infty - Y_{t-})dX_t \tag{6}$$

with initial condition $L_0 = 0$, where X_t is a subordinator.

If the process X cannot make jumps larger than 1 (which is natural to assume in this context), then the solution of this equation is

$$L_t = L_\infty(1 - \mathcal{E}_t(-X)), \tag{7}$$

where $\mathcal{E}_t(-X)$ denotes the stochastic exponential of the process $-X$ and the extreme length L_∞ is assumed to be constant. Observe that the process L_t defined by (7) is non-decreasing, and it reduces to the classical von Bertalanffy growth curve when X is a deterministic subordinator of the form $X_t = kt$.

This approach, like all existing ones, has a drawback as a growth model, since the asymptotic fish length is assumed to be constant. This implies that the variability in fish length and weight tends to vanish over time, which is unrealistic, as it would require all individuals to converge to the same limiting size.

To overcome this limitation, it is natural to assume that the extremal size of a fish is itself a random variable, thereby accounting for individual variability. It is therefore appropriate to employ backward stochastic differential equations (BSDEs), rather than forward SDEs, with a random terminal condition equal to the asymptotic weight (or length) of a fish.

We shall generalize expressions (3), (5) and (7) assuming that W_∞ is random variable and consider this variable as a boundary condition at infinity of a BSDE for W_t driven by a subordinator X and a Brownian Motion B, independent of X. The linear BSDEs derived in the paper differs from classical cases (see, e.g. [6], [4]) by considering not integrable coefficients on the infinite time interval. We assume that the extreme weight W_∞ (and size L_∞) of a fish is a random variable measurable with respect to the σ-algebra F^B_∞ generated by the Brownian Motion B, i.e., that two sources of randomness, the random individual variability (related with W_∞ or L_∞) and the environmental randomness (related with the process X_t), are independent. Under this assumption the BSDE takes simpler and more natural form (see Theorem 1 and 2).

2 The Fish weight growth and BSDEs

In this section, we extend the fish growth deterministic model (3) (the generalized von Bertalanffy model) to consider a random extremal weight and discuss the appropriate form of the BSDEs corresponding to the model (3) when W_∞ is a random variable.

We first assume the asymptotic weight W_∞ to be an integrable random variable and to depend on a Brownian motion, that represents the stochasticity in the model. In mathematical terms, we suppose W_∞ is measurable with respect to $F_\infty^B = \vee_{t\geq 0} F_t^B$, where B is a Brownian motion and $(F_t^B, t \geq 0)$ the related filtration.

Now we construct a simple growth model with a linear dynamics which is a solution of a BSDE. Assume that weight at-age process Y_t satisfies the linear backward equation

$$Y_t = \int_{t_0}^{t} Y_s f(s) ds + \int_{t_0}^{t} Z_s dB_s, \tag{8}$$

with the boundary condition at infinity

$$Y_\infty = \lim_{t\to\infty} Y_t = W_\infty. \tag{9}$$

Suppose that f is a deterministic function that regulates weight growth. If we require the average weight to follow the generalized von Bertalanffy model, then the function $f(t)$ will be equal to

$$\frac{\alpha p e^{-\alpha(t-t_0)}}{1 - e^{-\alpha(t-t_0)}}.$$

In fact, rewriting Equation (8) in the form

$$Y_t = W_\infty - \int_t^\infty Y_s f(s) ds - \int_t^\infty Z_s dB_s. \tag{10}$$

and considering the expected values, we observe that expected weight-at-age process satisfies the deterministic equation.

$$EY_t = EW_\infty - \int_t^\infty EY_s f(s) ds, \tag{11}$$

whose solution is

$$EY_t = EW_\infty \exp\{-\int_t^\infty f(s) ds\}. \tag{12}$$

Therefore, comparing with the von Bertlanffy model (3), we receive

$$EW_\infty \exp\{-\int_t^\infty f(s) ds\} = EW_\infty (1 - e^{-\alpha(t-t_0)})^p, \tag{13}$$

which implies

$$-\int_t^\infty f(s)ds = p\ln(1 - e^{-\alpha(t-t_0)}).$$

Differentiating we immediately get

$$f(t) = \frac{\alpha p e^{-\alpha(t-t_0)}}{1 - e^{-\alpha(t-t_0)}}.$$

Thus, the stochastic growth model

$$Y_t = \int_{t_0}^t Y_s \frac{p\alpha e^{-\alpha(s-t_0)}}{1 - e^{-\alpha(s-t_0)}} ds + \int_{t_0}^t Z_s dB_s, \tag{14}$$

with the boundary condition

$$Y_\infty = \lim_{t\to\infty} Y_t = W_\infty \tag{15}$$

can be interpreted as a backward extension of the von Bertalanffy model.

Moreover, its unique solution can be written explicitly as

$$W_t = E(W_\infty | F_t^B)(1 - e^{-\alpha(t-t_0)})^p. \tag{16}$$

More exactly, the solution of the BSDE (14)-(15), which characterizes the growth model, is the pair (Y_t, Z_t) given by

$$Y_t = W_t, \qquad Z_t = \varphi_t(1 - e^{-\alpha(t-t_0)})^p,$$

where W_t is defined by (16) and φ_t is identified via the martingale representation

$$E(W_\infty | F_t^B) = EW_\infty + \int_0^t \varphi_s dB_s.$$

Remark that even though the coefficient $\alpha p e^{-\alpha s}/(1 - e^{-\alpha s})$ in equation (14) is not integrable, the unique solution Y ensures that the product $Y_s \alpha p e^{-\alpha(s-t_0)}/(1 - e^{-\alpha(s-t_0)})$ is integrable.

Let $\widetilde{B}_t$ be a Brownian Motion independent of B and let $F_t = F_t^{B,\widetilde{B}}$ be the filtration generated by B and $\widetilde{B}$.

By analogy of (2) and expression (5) of Lv and Pitchford [3], let us define the process

$$W_t = W_\infty \left(1 - \mathcal{E}_t(-\alpha s + \sigma \widetilde{B})\right)^p \tag{17}$$

and consider this process as a model of weight at age evolution of a fish. For simplicity we take here $t_0 = 0$.

We assume that W_∞ is a random variable measurable with respect to $F^B_\infty = \vee_{t\geq 0} F^B_t$. The possible realization due to genetic/environmental variability is mathematically represented by the information flow $F_t = F_t^{B,\tilde{B}}$ and therefore the random variable that describes the weight at age W_t should be measurable with respect to F_t. That is, knowing the history up to t we know the random variable W_t. Mathematically, this will imply, from (17), that

$$W_t = E(W_\infty|F_t)(1 - \mathcal{E}_t\left(-\alpha s + \sigma B\right))^p . \tag{18}$$

Recalling the independence of B and $\tilde{B}$ and that W_∞ is F^B_∞ -measurable, we have that $E(W_\infty|F_t) = E(W_\infty|F^B_t)$ and

$$W_t = E(W_\infty|F^B_t)\left(1 - \mathcal{E}_t(-\alpha s + \sigma \widetilde{B})\right)^p . \tag{19}$$

Denote by $\mathcal{Y}$ a class of càdlàg processes $(Y_t, t \geq 0)$, for which the family $(Y_\tau, \tau \in \mathcal{T})$ is uniformly integrable, where $\mathcal{T}$ is the set of stopping times.

Equation (19) characterizes the growth model, which can be represented as the unique solution to a specific BSDE given in the following theorem.

Theorem 1 The process W_t from (19) is the unique solution, in the class $\mathcal{Y}$, of the following linear backward stochastic differential equation:

$$\begin{cases} Y_t = \frac{p}{2}\int_0^t (2\alpha H_s + \sigma^2(p-1)H_s^2)Y_s ds + \int_0^t Z_s^B dB_s + \int_0^t Z_s^{\tilde{B}} d\tilde{B}_s \\ Y_\infty = W_\infty \end{cases} \tag{20}$$

where the process H_s takes the form $H_s = \mathcal{E}_s(-\alpha u + \sigma\tilde{B})/(1 - \mathcal{E}_s(-\alpha u + \sigma\tilde{B}))$.

Note that if in (20) we set $p = 1$ and suppose the final condition to be constant, then the BSDE (20) can be written as

$$Y_t = \int_0^t Y_s \frac{\alpha\mathcal{E}(-\alpha s + \sigma\widetilde{B}_s)}{1 - \mathcal{E}(-\alpha s + \sigma\widetilde{B}_s)} ds + \int_0^t Z_s d\tilde{B}_s, \quad Y_\infty = L_\infty \tag{21}$$

which admits the same unique solution $L_t = L_\infty(1 - \mathcal{E}_t(-\alpha s + \sigma\widetilde{B}))$ as the forward SDE (4).

The backward generalization of the stochastic model (4) of Lv and Pitchford [3], when L_∞ is a random variable measurable with respect to F^B_∞, where B is a Brownian motion independent of $\widetilde{B}$, is the following BSDE

$$Y_t = \int_0^t Y_s \frac{\alpha \mathcal{E}(-\alpha s + \sigma \widetilde{B}_s)}{1 - \mathcal{E}(-\alpha s + \sigma \widetilde{B}_s)} ds + \int_0^t Z_s dB_s + \int_0^t \widetilde{Z}_s d\widetilde{B}_s, \tag{22}$$

with the boundary condition $Y_\infty = \lim_{t\to\infty} Y_t = L_\infty$. The unique solution of (22) is

$$L_t = E(L_\infty | F^B_t)(1 - \mathcal{E}_t(-\alpha s + \sigma \widetilde{B})),$$

which follows from Theorem 1 taking $p = 1$.

The mathematical expectation of L_t follows again the von Bertalanffy growth curve, since by independence of B and W

$$EL_t = E\left(E(L_\infty|F^B_t)(1 - \mathcal{E}_t(-\alpha s + \sigma \widetilde{B}))\right) =$$

$$= E(L_\infty)E(1 - \mathcal{E}_t(-\alpha s + \sigma \widetilde{B})) = EL_\infty(1 - E\mathcal{E}_t(-\alpha s + \sigma \widetilde{B})) = EL_\infty(1 - e^{-\alpha t}).$$

In the last result, we state the extension of the generalized model for the weight at age in the model considered by Russo et al. [8] omitting the proof. To this end, we introduce a subordinator

$$X_t = \alpha t + \int_0^t \int_{R_+} x \mu(ds, dx) \tag{23}$$

with (positive) jumps less that 1 and denote by μ the measure associated to the jumps of the process and ν its compensator. Moreover, B are independent Brownian motions also independent of the subordinator X.

Theorem 2 Let X be the subordinator (23) and W_∞ an integrable $\mathcal{F}^B$-measurable random variable. Then the process $Y_t = E[W_\infty|\mathcal{F}^B_t]\big(1 - \mathcal{E}_t(-X)\big)^p$ is the unique solution, in the class $\mathcal{Y}$, of the following linear backward stochastic differential equation

$$\begin{aligned} Y_t =& \alpha p \int_0^t Y_{s-} \frac{\mathcal{E}_{s-}(-X)}{1 - \mathcal{E}_{s-}(-X)} ds + \int_0^t \int_0^1 Y_{s-} \big[(1 + H(s,x))^p - 1\big] ds d\nu + \\ &+ \int_0^t Z_s d\widetilde{B}_s + \int_0^t \int_0^1 K(s,x) d(\mu - \nu), \end{aligned} \tag{24}$$

$$Y_\infty = W_\infty, \tag{25}$$

where $H(s,x) = x\mathcal{E}_{s-}(-X)/(1-\mathcal{E}_{s-}(-X))$.

Note that if X_t is a deterministic subordinator, i.e. if $X_t = \alpha t, \alpha > 0$, then equation (24) coincides with equation (14) and if $p = 1$, the equation (24)-(25) will be reduced to the BSDE

$$Y_t = (\alpha + m)\int_0^t Y_{s-}\frac{\mathcal{E}_{s-}(-X)}{1-\mathcal{E}_{s-}(-X)}ds + \int_0^t Z_s d\widetilde{B}_s +$$

$$+\int_0^t\int_{R_+} K(s,x)(\mu - \tilde{\mu})(dx, ds), \tag{26}$$

$$Y_\infty = \lim_{t\to\infty} Y_t = L_\infty \tag{27}$$

for the length process, with the unique solution equal to

$$L_t = E[L_\infty|\mathcal{F}_t^B]\big(1 - \mathcal{E}_t(-X)\big),$$

where $m \equiv \int_{R_+} x\nu(dx)$ is the mean jump size of the subordinator X.

The main objective of this work is theoretical, but we think that the models developed can be of interest for practical applications. The main novelty is the use of Backward instead of Forward SDEs, which is due to the assumption of the randomness of the extreme length of the fish. Our growth model differs from the one proposed by Russo et al [8] in that the extreme length L_∞ is replaced by its conditional mathematical expectation $E(L_\infty|F_t)$ and coincides with the model of Russo et al [8] if the terminal length is a constant. Since $E(L_\infty|F_t)$ is the best estimation of L_∞ in mean square sense, we expect that this model will be also in accordance with the real data. Of course, to this end the determination of the appropriate subordinator (characterized by many very small jumps, e.g., the gamma process) and selection of a suitable distribution for the extreme length is necessary.

The BSDE method requiers the extreme length or weight distribution as the input data to realize simulations backward in time. This method is applied if we know or can estimate the distribution of extreme fish length. The distribution of an extreme fish length is often easy to establish, because extreme value theory provides general limiting forms, extreme value distributions (e.g., Weibul, Gumbel, Frechet distributions). This means that one can often approximate the distribution of the maximum without knowing full details of the underlying dynamics. Then it will be possible to determine the conditional expectations and the fish length at previous time moments by simulations backward in time.

References

[1] Garcıa-Rodriguez, M., Pereda, P., Landa, J., Esteban, A., 2005. On the biology growth of the anglerfish lophius Budegasa Spinola, 1807 in the Spanish Mediterranean: a preliminary approach. Fisheries research 71(2), 197–208.

[2] G. Gudmundsson, Stochastic growth. Can. J. Fish. Aquat. Sci. 62, pp. 1746–1755.

[3] Q. Lv and J. W. Pitchford, Stochastic von Bertalanffy models with applications to fish recruitment. J. Theoret. Biol. 244(4), 2007, pp. 640–655.

[4] A. Papapantoleon, D. Possamaï and A. Saplaouras, Existence and uniqueness results for BSDE with jumps: the whole nine yards, Electron. J. Probab. 23 , no. 121, 2018, pp. 1–68.

[5] D. Pauly, Gill size and temperature as governing factors in fish growth: A generalization of von Bertalanffy's growth formula (2nd edition). Fisheries Centre Research Reports 32(2), 2024, 105 pp.

[6] M. Quenez and A. Sulem. BSDEs with jumps, optimization and applications to dynamic risk measures. Stochastic Processes and their Applications, 123(8), pp.3328-3357, 2013.

[7] P. Román-Román, D. Desirée Romero, F. Torres-Ruiz. A diffusion process to model generalized von bertalanffy growth patterns: Fitting to real data. Journal of Theoretical Biology, 2010, 263 (1), pp.59.

[8] T. Russo, P. Baldi, A. Parisi, G. Magnifico, S. Mariani and S. Cataudella, Lévy processes and stochastic von Bertalanffy models of growth, with application to fish population analysis, Journal of Theoretical Biology Volume 258, Issue 4, 21 June 2009, Pages 521-529

[9] L. Von Bertalanffy, A quantitative theory of organic growth. Human Biol. 10, 1938, pp. 181–213.