On Sub-Gaussianity in Banach Spaces

George Giorgobiani, Vakhtang Kvaratskhelia and Vazha Tarieladze
Muskhelishvili Institute of Computational Mathematics
of the Georgian Technical University,
Tbilisi, Georgia
emails: giorgobiani.g@gtu.ge, v.kvaratskhelia@gtu.ge, v.tarieladze@gtu.ge

Abstract

We show that if X is a Banach space and a weakly sub-Gaussian random element in X induces the 2-summing operator, then it is $T-$sub-Gaussian provided that X is a reflexive type 2 space. Using this result we obtain a characterization of weakly sub-Gaussian random elements in a Hilbert space which are $T-$sub-Gaussian.

***Keywords*— Sub-Gaussian random variable, Gaussian random variable, weakly sub-Gaussian random element, $T-$sub-Gaussian random element, Banach space, Hilbert space.**

I. Introduction

Let $(\Omega, \mathcal{A}, \mathbf{P})$ be a probability space. Following [8] (see also [3], [9], [15]) a real-valued random variable $\xi : \Omega \to \mathbb{R}$ is **sub-Gaussian** if there exists a real number $a \geq 0$ such that for every real number t the following inequality holds:

$$\mathbb{E}\, e^{t\xi} \leq e^{\frac{1}{2}a^2t^2},$$

where $\mathbb{E}$ stands for the mathematical expectation.

To each random variable ξ it corresponds a parameter $\tau(\xi) \in [0, +\infty]$ defined as follows (we agree $\inf(\emptyset) = +\infty$):

$$\tau(\xi) = \inf\left\{a \geq 0 : \quad \mathbb{E}\, e^{t\xi} \leq e^{\frac{1}{2}a^2t^2}, \quad \forall t \in \mathbb{R}\right\}.$$

A random variable ξ is sub-Gaussian if and only if $\tau(\xi) < +\infty$ and $\mathbb{E}\xi = 0$. Moreover, if ξ is a sub-Gaussian random variable, then for every real number t

$$\mathbb{E}\, e^{t\xi} \leq e^{\frac{1}{2}\tau^2(\xi)t^2}$$

and

$$\left(\mathbb{E}\xi^2\right)^{\frac{1}{2}} \leq \tau(\xi)\,.$$

If ξ is a Gaussian random variable with $\mathbb{E}\xi = 0$, then ξ is sub-Gaussian and

$$\left(\mathbb{E}\xi^2\right)^{\frac{1}{2}} = \tau(\xi)\,.$$

Remark 1.1: [3, Example 1.2]. If ξ is a bounded random variable, i.e. if for some constant $c \in \mathbb{R}$ with $0 \leq c < +\infty$, we have $|\xi| \leq c$ a.s. and $\mathbb{E}\xi = 0$, then ξ is sub-Gaussian and $\tau(\xi) \leq c$.

Denote by $\mathcal{SG}(\Omega, \mathcal{A}, \mathbb{P})$, or in short, by $\mathcal{SG}(\Omega)$ the set of all sub-Gaussian random variables defined on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$. It is a remarkable fact that $\mathcal{SG}(\Omega)$ *is a vector space over* $\mathbb{R}$ with respect to the natural point-wise operations; moreover, the functional $\tau(\cdot)$ is a norm on $\mathcal{SG}(\Omega)$ (provided that random variables which coincide almost surely are identified) and $(\mathcal{SG}(\Omega), \tau(\cdot))$ is a Banach space [2]. For $\xi \in \mathcal{SG}(\Omega)$ instead of $\tau(\xi)$ we will write also $\|\xi\|_{\mathcal{SG}(\Omega)}$.

More information about the sub-Gaussian random variables can be found for example in [6].

Remark 1.2: [3, Theorem 1.3] (see also [15, Proposition 2.9]). For a sub-Gaussian random variable ξ we have

$$\vartheta(\xi) = \sup_{n \geq 1} \frac{\left(\mathbb{E}\,\xi^{2n}\right)^{1/2n}}{n^{1/2}} < +\infty,$$

the functional ϑ is a norm on the vector space $\mathcal{SG}(\Omega)$ and the norms τ and ϑ are equivalent, i.e. there exist positive constants a_1 and a_2 such that for every $\xi \in \mathcal{SG}(\Omega)$ we have

$$a_1\vartheta(\xi) \leq \tau(\xi) \leq a_2\vartheta(\xi).$$

In an infinite dimensional Banach space there are several notions of sub-Gaussianity. The aim of the paper is to show that these concepts are different in general. We also give some sufficient conditions for their equivalence.

Let X be a Banach space over $\mathbb{R}$ with a norm $\|\cdot\|$ and X^* be its dual space. The value of the linear functional $x^* \in X^*$ at an element $x \in X$ is denoted by the symbol $\langle x^*, x\rangle$.

Following [16, p. 88] a mapping $\xi : \Omega \to X$ is called a random element (vector) in X if $\langle x^*, \xi\rangle$ is a random variable for every $x^* \in X^*$.

If $0 < p < \infty$, then a random element ξ in a Banach space X:

- has *a strong p-th order*, if $\|\xi\|$ is a random variable and $\mathbb{E}\,\|\xi\|^p < \infty$;
- has *a weak p-th order*, if $\mathbb{E}\,|\langle x^*, \xi\rangle|^p < \infty$ for every $x^* \in X^*$;
- is *centered*, if ξ has a weak first order and $\mathbb{E}\,\langle x^*, \xi\rangle = 0$ for every $x^* \in X^*$.

To each weak second-order centered random element ξ in a separable Banach space X it corresponds a mapping $R_\xi : X^* \to X$ such that

$$\langle y^*, R_\xi x^*\rangle = \mathbb{E}\,\langle y^*, \xi\rangle\langle x^*, \xi\rangle, \quad \text{for every} \quad x^*, y^* \in X^*,$$

which is called *the covariance operator of* ξ [16, Corollary 2 (p.172)].

A random element $\xi : \Omega \to X$ is called *Gaussian*, if for each functional $x^* \in X^*$ the random variable $\langle x^*, \xi\rangle$ is Gaussian.

A mapping $R : X^* \to X$ is said to be *a Gaussian covariance*, if there exists a Gaussian random element in X whose covariance operator is R.

A random element $\xi : \Omega \to X$ will be called *weakly sub-Gaussian* [14], if for each $x^* \in X^*$ the random variable $\langle x^*, \xi\rangle$ is sub-Gaussian.

A random element $\xi : \Omega \to X$ will be called *$T-$sub-Gaussian* (or $\gamma-$sub-Gaussian [5]), if there exists a probability space $(\Omega', \mathcal{A}', \mathbf{P}')$ and a centered Gaussian random element $\eta : \Omega' \to X$ such that for each $x^* \in X^*$

$$\mathbb{E}\, e^{\langle x^*, \xi\rangle} \le \mathbb{E}\, e^{\langle x^*, \eta\rangle}\,. \tag{1.1}$$

Theorem 1.3: (a) If X is finite-dimensional Banach space, then every weakly sub-Gaussian random element in X is $T-$sub-Gaussian.

(b) If X is infinite-dimensional separable Banach space, then there exist a weakly sub-Gaussian random element in X. which *is not* $T-$sub-Gaussian.

Proof: (a). See [15, Proposition 4.9].

(b). According to [14] (see also [15, Theorem 4.5]) we can find and fix a weakly sub-Gaussian random element ξ in X, such that $\mathbb{E}\|\xi\| = \infty$. Such a random element cannot be $T-$sub-Gaussian, because according to a remarkable [5, Theorem 3.4] every such random element must 'exponentially integrable'!

To every weakly sub-Gaussian random element $\xi : \Omega \to X$ we associate *the induced linear operator*

$$T_\xi : X^* \to \mathcal{SG}(\Omega)$$

defined by the equality:

$$T_\xi x^* = \langle x^*, \xi\rangle \qquad \text{for all} \quad x^* \in X^*.$$

Let X and Y be Banach spaces, $L(X, Y)$ be the space of all continuous linear operators acting from X to Y. An operator $T \in L(X, Y)$ is called 2-(absolutely) summing if there exists a constant $C > 0$ such that for each natural number n and for every choice $x_1, x_2, \dots, x_n$ of elements from X we have

$$\left(\sum_{k=1}^{n} ||Tx_k||^2\right)^{1/2} \le C \sup_{||x^*||_{X^*} \le 1} \left(\sum_{k=1}^{n} |\langle x^*, x_k\rangle|^2\right)^{1/2}. \tag{1.2}$$

For a 2-summing $T : X \to Y$ we denote the minimum possible constant C in (1.2) by $\pi_2(T)$.

We say that a Banach space X has type 2, if there exists a finite constant $C \ge 0$ such that for each natural number n and for every choice $x_1, x_2, \dots, x_n$ of elements from X we have

$$\left(\int_0^1 \left\|\sum_{k=1}^{n} r_k(t)x_k\right\|^2 dt\right)^{1/2} \le C \left(\sum_{k=1}^{n} \|x_k\|^2\right)^{1/2},$$

where $r_1(\cdot), \dots, r_n(\cdot)$ are Rademacher functions on $[0, 1]$. An example of a type 2 space is a Hilbert space as well as the spaces $l_p, L_p([0, 1]), 2 \le p < +\infty$.

II. Main results

The following theorem is a slightly corrected version of [10, Theorem 1.7].

Theorem 2.1: Let X be a separable Banach space. For a *weakly sub-Gaussian* random element $\xi : \Omega \to X$ consider the assertions:

(i) ξ is $T-$sub-Gaussian.

(ii) $T_\xi : X^* \to \mathcal{SG}(\Omega)$ is a 2-summing operator.

Then:

(a) $(i) \Longrightarrow (ii)$;

(b) The implication $(ii) \Longrightarrow (i)$ is true provided that X is a reflexive Banach space of type 2.

Proof: (a) (i) implies that there exists a centered Gaussian random element $\eta : \Omega' \to X$ such that for each $x^* \in X^*$ the relation (1.1) holds. This implies that

$$\tau(T_\xi x^*) \le \tau(T_\eta x^*) \qquad \text{for all} \quad x^* \in X^* .$$

Thus, as η is a Gaussian random element in X, the operator T_η is 2-summing (see, for example, [4]). Hence, we conclude that (ii) holds.

(b) Since ξ is a weakly sub-Gaussian random element, for every $x^* \in X^*$ we can write:

$$\mathbb{E}\, e^{\langle x^*, \xi\rangle} \le e^{\frac{1}{2}||T_\xi x^*||^2_{\mathcal{SG}(\Omega)}}.$$

Taking into account that the operator T_ξ is 2-summing and X is reflexive, by Pietsch domination theorem (see [11] or [16, Theorem 2.2.2]), there exists a probability measure μ defined on the $\sigma(X, X^*)$-Borel sigma-algebra of the unit ball $B_X \subset X$ such that

$$||T_\xi x^*||^2_{\mathcal{SG}(\Omega)} \le \pi_2^2(T_\xi) \int\limits_{B_X} \langle x^*, x\rangle^2 \, \mu(dx), \quad x^* \in X^* .$$

If we consider μ as a probability measure in X concentrated on B_X, then for every $x^* \in X^*$

$$\int\limits_{B_X} \langle x^*, x\rangle^2 \, \mu(dx) = \int\limits_{X} \langle x^*, x\rangle^2 \, \mu(dx) = \langle R_\mu x^*, x^*\rangle,$$

where R_μ is the covariance operator of μ. As μ is concentrated on the bounded set, it clearly has a strong second order, and taking into account the fact that X is a type 2 space, we obtain that R_μ is a Gaussian covariance (see [4, Theorem 3.1]). Denoting $\pi_2^2(T)R_\mu = R$, we get

$$\mathbb{E}\, e^{\langle x^*, \xi\rangle} \le e^{\frac{1}{2}\langle R x^*, x^*\rangle}, \qquad x^* \in X^*,$$

and, thus, ξ is a $T-$sub-Gaussian random element as R is a Gaussian covariance. $\square$

Problem 2.2: Prove that the reflexivity condition for X in Theorem 2.1(b) can be removed.

Consider now the case when $X = H$, where H denotes an infinite-dimensional separable Hilbert space with the inner product $\langle \cdot, \cdot \rangle$. As usual we identify H^* with H by means of the equality $H^* = \{\langle \cdot, y\rangle : y \in H\}$.

From Theorem 2.1 we will derive now the following result, which is related with the similar assertion contained in [1, Proposition 3.1].

Theorem 2.3: Let H be an infinite-dimensional separable Hilbert space. For a weakly sub-Gaussian random element $\xi : \Omega \to H$ the following statements are equivalent:

(i) ξ is $T-$sub-Gaussian.

(ii_m) For each orthonormal basis (φ_k) of H

$$\sum_{k=1}^{\infty} \tau^2(\langle \varphi_k, \xi\rangle) < \infty . \tag{2.1}$$

Proof: The implication $(i) \Longrightarrow (ii_m)$ follows from Theorem 2.1 (a).

The implication $(ii_m) \Longrightarrow (i)$ follows from Theorem 2.1 (b) as H is a type 2 space and according to [12] the condition (ii_m) implies that the condition (ii) of Theorem 2.1 is satisfied as well. $\square$

In connection with Theorem 2.3 naturally arises the following question: is it possible to replace the condition (ii_m) by the following (weaker) condition?

(ii_w) *There is an orthonormal basis* (φ_k) *of* H *such that*

$$\sum_{k=1}^{\infty} \tau^2(\langle \varphi_k, \xi \rangle) < \infty \,.$$

In [1, Remark 4.3] it is claimed that the answer to this question *is positive*.

At the end we pose another interesting question related to Theorem 2.3: does there exist a bounded centered random element ξ in a separable infinite-dimensional Hilbert space H such that

$$\sum_{k=1}^{\infty} \tau^2(\langle \psi_k, \xi \rangle) = \infty$$

for *every* orthonormal bases (ψ_k) of H?

Conclusion

We have shown that in an infinite dimensional Banach space, the notions of week sub-Gaussianity and T-sub-Gaussianity do not coincide. Sufficient conditions for their equivalence in a general, infinite dimensional Banach space is given in terms of 2-summing induced operators.

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