Optimal Mean-Variance Robust Hedging

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Abstract

We consider financial market with yield process of risky asset satisfying the so-called structure condition and construct optimal mean-variance robust hedging strategy for misspecified asset price process. In particular, we study the stochastic volatility process with fully defined volatility process with small randomness and misspecified asset price process.

Key words and phrases: misspecified asset price process, stochastic volatility process with small randomness.

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1 A financial market model

Let $(\Omega, \mathcal{F}, F) = (\mathcal{F}_t)_{0\le t\le T}, P)$ be a filtered probability space with filtration F satisfying the usual conditions, where $T \in (0,\infty]$ is a fixed time horizon. Assume that $\mathcal{F}_0$ is trivial and $\mathcal{F}_T = \mathcal{F}$.

There exist $d+1$, $d \ge 1$, primitive assets: one bond, whose price process is assumed to be 1 at all times and d risky assets (stocks), whose R^d-valued price process $X = (X_t)_{0\le t\le T}$ is a continuous semimartingale given by the relation

$$dX_t = \operatorname{diag}(X_t)dR_t, \quad X_0 > 0, \tag{1.1}$$

where $\operatorname{diag}(X)$ denotes the diagonal $d \times d$-matrix with diagonal elements $X^1, \dots, X^d$, and the yield process $R = (R_t)_{0 \le t \le T}$ is a R^d-valued continuous semimartingale satisfying the structure condition (SC). That is (see Schweizer [6]),

$$dR_t = d\langle \widetilde{M} \rangle_t \lambda_t + d\widetilde{M}_t, \quad R_0 = 0, \tag{1.2}$$

where $\widetilde{M} = (\widetilde{M}_t)_{0 \le t \le T}$ is a R^d-valued continuous martingale, $\widetilde{M} \in \mathcal{M}^2_{0,loc}(P)$, $\lambda = (\lambda_t)_{0 \le t \le T}$ is a F-predictable R^d-valued process, and the mean-variance tradeoff (MVT) process $\widetilde{\mathcal{K}} = (\widetilde{\mathcal{K}}_t)_{0 \le t \le T}$ of process R

$$\widetilde{\mathcal{K}}_t := \int_0^t \lambda'_s d\langle \widetilde{M} \rangle_s \lambda_s = \langle \lambda' \cdot \widetilde{M} \rangle_t < \infty \quad P\text{-a.s.}, \quad t \in [0, T]. \tag{1.3}$$

Remark 1. Remember that all vectors are assumed to be column vectors.

Suppose that the martingale $\widetilde{M}$ has the form

$$\widetilde{M} = \sigma \cdot M, \tag{1.4}$$

where $M = (M_t)_{0 \le t \le T}$ is a R^d-valued continuous martingale, $M \in \mathcal{M}^2_{0,loc}(P)$, $\sigma = (\sigma_t)_{0 \le t \le T}$ is a $d \times d$-matrix valued, F-predictable process with $\operatorname{rank}(\sigma_t) = d$ for any t, P-a.s., the process $(\sigma_t^{-1})_{0 \le t \le T}$ is locally bounded, and

$$\langle \widetilde{M} \rangle_T = \int_0^T \sigma_t d\langle M \rangle_t \sigma'_t < \infty, \quad P\text{-a.s.} \tag{1.5}$$

Assume now that the following condition is satisfied:

There exists fixed R^d-valued, F-predictable process $k = (k_t)_{0 \le t \le T}$ such that

$$\lambda = \lambda(\sigma) = (\sigma')^{-1} k. \tag{1.6}$$

In this case, from (1.2) we get

$$\begin{aligned} dR_t &= d\langle \widetilde{M} \rangle_t \lambda_t + d\widetilde{M}_t = \sigma_t d\langle M \rangle_t \sigma'_t (\sigma'_t)^{-1} k_t + \sigma_t dM_t \\ &= \sigma_t (d\langle M \rangle_t k_t + dM_t) \end{aligned} \tag{1.7}$$

and

$$\begin{aligned} \widetilde{\mathcal{K}}_t &= \int_0^t \lambda'_s d\langle \widetilde{M} \rangle_s \lambda_s = \int_0^t k'_t ((\sigma'_t)^{-1})' \sigma_t d\langle M \rangle_t \sigma'_t (\sigma'_t)^{-1} k_t \\ &= \int_0^t k'_t d\langle M \rangle_t k_t = \langle k \cdot M \rangle_t := \mathcal{K}_t. \end{aligned}$$

From (1.3) we have

$$\mathcal{K}_t < \infty, \quad P\text{-a.s. for all } t \in [0,T]. \tag{1.8}$$

Thus, if we introduce the process $M^0 = (M_t^0)_{0 \le t \le T}$ by the relation

$$dM_t^0 = d\langle M\rangle_t k_t + dM_t, \quad M_0^0 = 0, \tag{1.9}$$

then the MVT process $\mathcal{K} = (\mathcal{K}_t)_{0 \le t \le T}$ of R^d-valued semimartingale M^0 is finite, and hence M^0 satisfies SC.

Finally, the scheme (1.1), (1.2), (1.4), (1.6) and (1.9) can be rewritten in the following form:

$$\begin{aligned} dX_t &= \mathrm{diag}(X_t) dR_t, \quad X_0 > 0, \\ dR_t &= \sigma_t dM_t^0, \quad R_0 = 0, \\ dM_0^0 &= d\langle M\rangle_t k_t + dM_t, \quad M_0 = 0, \end{aligned} \tag{1.10}$$

where σ and k satisfy (1.5) and (1.8), respectively.

This is our financial market model.

2 Stochastic volatility process with small diffusion coefficient

Denote by $\mathrm{Ball}_L(0,r)$, $r \in [0,\infty)$, the closed r-radius ball in the space $L = L_\infty(dt \times dP)$, with the center at the origin, and let

$$\mathcal{H} := \big\{ h = \{h_{ij}\}, \;\; i,j, = \widehat{1,d} \; : \; h \text{ is } F\text{-predictable, } d \times d\text{-matrix valued process}, \;\; \mathrm{rank}(h) = d, \;\; h_{ij} \in \mathrm{Ball}_L(0,r), \;\; r \in [0,\infty) \big\}.$$

The class $\mathcal{H}$ is called the class of alternatives.

Fix the value of small parameter $\delta > 0$, as well as $d \times d$-matrix valued F-predictable process $\sigma^0 = (\sigma_t^0)_{0 \le t \le T}$, $\mathrm{rank}(\sigma^0) = d$, with

$$\int_0^T \sigma_t^0 d\langle M\rangle_t (\sigma_t^0)' < \infty \;\; P\text{-a.s.}$$

Denote

$$A_\delta = \{\sigma \; : \; \sigma = \sigma^0 + \delta h, \;\; h \in \mathcal{H}\}.$$

As an example, consider now particular case.

Let $a(t,y)$ be a drift coefficient of volatility process. Introduce the processes described by the following system of SDE:

$$\begin{aligned} dX_t &= X_t\, dR_t, \quad X_0 > 0, \\ dR_t &= (\sigma_t^0 + \delta h_t) dM_t^0, \quad R_0 = 0, \\ dY_t &= a(t, Y_t)dt + \varepsilon dw_t^\sigma, \quad Y_0 = 0, \quad 0 \le t \le T, \end{aligned} \tag{2.1}$$

where

$$dM_t^0 = k_t dt + dw_t^R,$$

$h \in \mathcal{H}$ and σ_t^0 is the center of the confidence interval of volatility, which shrinks to

$$\sigma_t = f^{\frac{1}{2}}(Y_t).$$

Here, $w = (w^R, w^\sigma)$ is a standard two-dimensional Wiener process, defined on complete probability space $(\Omega, \mathcal{F}, P)$, $\mathcal{F}^w = (\mathcal{F}_t^w)_{0\le t\le T}$ is the Paugmentation of the natural filtration $\mathcal{F}_t^w = \sigma(w_s, 0 \le s \le t)$, $0 \le t \le T$, generated by w, $f(\cdot)$ is a continuous one-to-one positive locally bounded function (e.g., $f(x) = e^x$). Assume that the system (2.1) has a unique strong solution.

As a result, we get the so-called stochastic volatility process with small randomness and misspecified asset price process.

3 Construction of optimal mean-variance robust hedging strategy

Consider the set of processes $\{R^\sigma$ (or X^σ), $\sigma \in A_\delta\}$, which represents the misspecification of asset price process.

Define the class of admissible trading strategies $\Theta = \Theta(\sigma^0)$.

Definition 1. The class $\Theta = \Theta(\sigma^0)$ is a class of R^d-valued F-predictable processes $\theta = (\theta_t)_{0\le t\le T}$ such that

$$E\int_0^T \theta_t' \sigma_t^0 d\langle M\rangle_t (\sigma_t^0)'\theta_t < \infty, \quad E\int_0^T \theta_t' d\langle M\rangle_t \theta_t < \infty. \tag{3.1}$$

Let $\theta \in \Theta$ be the dollar amount (rather than the number of shares) invested in the stock X^σ, $\sigma \in A_\delta$. Then, for each $\sigma \in A_\delta$, the trading gains induced by the self-financing portfolio strategy associated to θ has the form

$$G_t(\sigma, \theta) = \int_0^t \theta'_s dR^\sigma_s, \quad 0 \le t \le T, \tag{3.2}$$

where $R^\sigma = (R^\sigma_t)_{0 \le t \le T}$ is the yield process given by (1.10).

Introduce the notation

$$\mathcal{M}^e_2 := \Big\{ Q \sim P \ : \ \frac{dQ}{dP} \in L^2(P), \ \ M^0 \text{ is a } Q\text{-local martingale} \Big\},$$

and suppose that

$$(c.1) \qquad\qquad \mathcal{M}^e_2 \neq \emptyset.$$

Introduce the condition:

(c.2) There exists equivalent local martingale measure (ELMM) $\overline{Q}$, such that the density process $z = z^{\overline{Q}}$ satisfies the reverse Hölder inequality $R_2(P)$, see definition in [4].

It is well-known, that under the conditions (c.1) and (c.2) the density process $\widetilde{z} = (\widetilde{z}_t)_{0 \le t \le T}$ of the variance-optimal ELMM satisfies $R_2(P)$ as well, see Delbaen et al. [1].

Now under the conditions (c.1) and (c.2) the r.v. $G_T(\sigma, \theta) \in L^2(P)$, $\forall \sigma \in A_\delta$, and the space $G_T(\sigma, \Theta)$ is closed in $L^2(P)$, $\forall \sigma \in A_\delta$ (see, e.g., Theorem 2 of Rheinländer and Schweizer [4]).

Remark 2. 1. Condition $E \int_0^T \theta'_t d\langle M\rangle_t \theta_t < \infty$ from (3.1) is equivalent to the condition $E \int_0^T \theta'_t h_t d\langle M\rangle_t h'_t \theta_t < \infty$, $\forall h_t \in \mathcal{H}$, since each component (h_{ij}) of matrix h is bounded (by r), and $\mathcal{H}$ contains the constants.

2. Under conditions (c.1) and (c.2),

$$E\bigg(\int_0^T |\theta'_t \sigma^0_t d\langle M\rangle_t k_t| \bigg)^2 \le \text{const.}\, E \int_0^T \theta'_t \sigma^0_t d\langle M\rangle_t (\sigma^0_t)' \theta_t$$

and

$$\begin{aligned} E\bigg(\int_0^T |\theta'_t h_t d\langle M\rangle_t k_t| \bigg)^2 &\le \text{const.}\, E \int_0^T \theta'_t h_t d\langle M\rangle_t h'_t \theta_t \\ &\le \text{const.}\, E \int_0^T \theta'_t d\langle M\rangle_t \theta_t, \quad \forall h_t \in \mathcal{H}, \end{aligned}$$

as it follows from above mentioned Theorem 2 of [4] (namely, from the equality $\Theta := L^2(M) \cap L^2(A) = L^2(M)$) and definition of class $\mathcal{H}$.

A contingent claim is an $\mathcal{F}_T$-measurable square-integrable r.v. H, which models the payoff from a financial product at the maturity date T.

The problem we are interested in is to find the robust hedging strategy for a contingent claim H in the above described incomplete financial market model with misspecified asset price process X^σ, $\sigma \in A_\delta$, using mean-variance approach.

For each $\sigma \in A_\delta$, the total loss of a hedger, who starts with the initial capital x, uses the strategy θ, believes that the stock price process follows X^σ, and has to pay a random amount H at the date T, is $H - x - G_T(\sigma, \theta)$.

Denote

$$\mathcal{J}(\sigma, \theta) := E(H - x - G_T(\sigma, \theta))^2. \tag{3.3}$$

One setting of the robust mean-variance hedging problem consist in solving the optimization problem

$$\text{minimize} \sup_{\sigma \in A_\delta} \mathcal{J}(\sigma, \theta) \text{ over all strategies } \theta \in \Theta. \tag{3.4}$$

We "slightly" change this problem using the approach developed in Toronjadze [7] which based on the following approximation

$$\begin{aligned}\sup_{\sigma \in A_\delta} \mathcal{J}(\sigma, \theta) &= \exp\left\{ \sup_{h \in \mathcal{H}} \mathcal{J}(\sigma^0 + \delta h, \theta) \right\} \\ &\simeq \left\{ \sup_{h \in \mathcal{H}} \left[\ln \mathcal{J}(\sigma^0, \theta) + \delta \, \frac{D\mathcal{J}(\sigma^0, h, \theta)}{\mathcal{J}(\sigma^0, \theta)} \right] \right\} \\ &= \mathcal{J}(\sigma^0, \theta) \exp\left\{ \delta \sup_{h \in \mathcal{H}} \frac{D\mathcal{J}(\sigma^0, h, \theta)}{\mathcal{J}(\sigma^0, \theta)} \right\},\end{aligned}$$

where

$$D\mathcal{J}(\sigma^0, h, \theta) := \frac{d}{d\delta} \mathcal{J}(\sigma^0 + \delta h, \theta)\Big|_{\delta=0} = \lim_{\delta \to 0} \frac{\mathcal{J}(\sigma^0 + \delta h, \theta) - \mathcal{J}(\sigma^0, \theta)}{\delta}$$

is the Gateaux differential of the functional $\mathcal{J}$ at the point σ^0 in the direction h.

Approximate (in leading order δ) the optimization problem (3.4) by the problem

$$\text{minimize } \mathcal{J}(\sigma^0, \theta) \exp\left\{ \delta \sup_{h \in \mathcal{H}} \frac{D\mathcal{J}(\sigma^0, h, \theta)}{\mathcal{J}(\sigma^0, \theta)} \right\} \quad \text{over all strategies } \theta \in \Theta. \tag{3.5}$$

Note that each solution θ^* of the problem (3.5) minimizes $\mathcal{J}(\sigma^0, \theta)$ under the constraint

$$\sup_{h\in\mathcal{H}} \frac{D\mathcal{J}(\sigma^0, h, \theta)}{\mathcal{J}(\sigma^0, \theta)} \le c := \sup_{h\in\mathcal{H}} \frac{D\mathcal{J}(\sigma^0, h, \theta^*)}{\mathcal{J}(\sigma^0, \theta^*)} .$$

This characterization of an optimal strategy θ^* of the problem (3.5) leads to the

Definition 2. The trading strategy $\theta^* \in \Theta$ is called optimal mean-variance robust trading strategy against the class of alternatives $\mathcal{H}$ if it is a solution of the optimization problem

$$\text{minimize } \mathcal{J}(\sigma^0, \theta) \text{ over all } \theta \in \Theta, \text{ subject to constraint}$$
$$\sup_{h\in\mathcal{H}} \frac{D\mathcal{J}(\sigma^0, h, \theta)}{\mathcal{J}(\sigma^0, \theta)} \le c, \tag{3.6}$$

where c is some generic constant.

Remark 3. In contrast to "mean-variance robust" traiding strategy, which associates with optimization problem (3.4) and control theory, we find the "optimal mean-variance robust" trading strategy in the sense of Definition 2. Such approach and term are common in robust statistics theory (see, e.g., Hampel et al. [3], Rieder [5]).

To solve the problem (3.6), we need to calculate $D\mathcal{J}(\sigma^0, h, \theta)$.

Following Rheinländer and Schweizer [4] and Gourieroux et al. [2], introduce the probability measure $\widetilde{Q} \sim P$ on $\mathcal{F}_T$ by the relation

$$d\widetilde{Q} = \frac{\widetilde{z}_T}{\widetilde{z}_0}\, d\widetilde{P} \quad (\text{and hence } d\widetilde{Q} = \frac{\widetilde{z}_T^2}{\widetilde{z}_0}\, dP). \tag{3.7}$$

Using Proposition 5.1 of Gourieroux et al. [2], we can write

$$\begin{aligned}
\mathcal{J}(\sigma, \theta) &= E\, \frac{\widetilde{z}_T^2}{\widetilde{z}_0^2}\, \frac{\widetilde{z}_0^2}{\widetilde{z}_T^2} \left(H - x - \int_0^T \theta_t' dR_t^\sigma \right)^2 \\
&= \widetilde{z}_0^{-1} E^{\widetilde{Q}}\, \frac{\widetilde{z}_0^2}{\widetilde{z}_T^2} \left(H - x - \int_0^T \theta_t' \sigma_t dM_t^0 \right)^2 \\
&= \widetilde{z}_0^{-1} E^{\widetilde{Q}} \left(\frac{H\widetilde{z}_0}{\widetilde{z}_T} - x - \int_0^T \psi_t^0(\sigma) d\frac{\widetilde{z}_0}{\widetilde{z}_t} - \int_0^T \big(\psi_t^1(\sigma)\big)' d\frac{M_t^0}{\widetilde{z}_t} \widetilde{z}_0 \right)^2 \\
&:= \mathcal{J}(\sigma, \psi^0, \psi^1),
\end{aligned} \tag{3.8}$$

where

$$\psi_t^1(\sigma)=\sigma_t'\theta_t,\quad \psi_t^0(\sigma)=x+\int_0^t\theta_s'\sigma_s dM_s^0-\theta_t'\sigma_t M_t^0,\quad 0\le t\le T. \tag{3.9}$$

Thus

$$\psi_t^1(\sigma)=\psi_t^1(\sigma^0)+\delta\psi_t^1(h),\quad \psi_t^0(\sigma)=\psi_t^0(\sigma^0)+\delta\overline{\psi}_t^0(h),$$

where

$$\overline{\psi}_t^0(h)=\psi_t^0(h)-x.$$

Let (following Rheinländer and Schweizer [4])

$$\frac{H}{\widetilde{z}_T}\widetilde{z}_0=E\biggl(\frac{H}{\widetilde{z}_T}\widetilde{z}_0\biggr)+\int_0^T(\psi_t^H)'dU_t+L_T \tag{3.10}$$

be the Galtchouk-Kunita-Watanabe decomposition of r.v. $\frac{H}{\widetilde{z}_T}\widetilde{z}_0$ w.r.t $R^{(d+1)}$-valued $\widetilde{Q}$-local martingale $U=(\frac{\widetilde{z}_0}{\widetilde{z}},\frac{M^0}{\widetilde{z}},\widetilde{z}_0)'$, where $\psi^H=(\psi^{0,H},\psi^{1,H})'\in L^2(U,\widetilde{Q})$, the space of F-predictable processes ψ such that $\int\psi' dU\in\mathcal{M}^2(\widetilde{Q})$ of martingales, and $L\in\mathcal{M}^2_{0,loc}(\widetilde{Q})$, L is $\widetilde{Q}$-strongly orthogonal to U.

Denote

$$\psi=(\psi^0,\psi^1)'\quad\text{and}\quad\overline{\psi}=(\overline{\psi}^0,\psi^1)'. \tag{3.11}$$

Then, using (3.8), (3.9) and (3.10) we can write for each h

$$\begin{aligned}
&\mathcal{J}(\sigma^0+\delta h,\psi)=\mathcal{J}(\sigma^0,\psi)+\delta\cdot 2\widetilde{z}_0^{-1}\\
&\times E^{\widetilde{Q}}\biggl\{\biggl[\biggl(x-E^{\widetilde{Q}}\frac{H}{\widetilde{z}_T}\widetilde{z}_0\biggr)-L_T+\int_0^T(\psi_t(\sigma^0)-\psi_t^H)'dU_t\biggr]\int_0^T(\overline{\psi}_t(h))'dU_t\biggr\}\\
&\qquad+\delta^2\widetilde{z}_0^{-1}E^{\widetilde{Q}}\biggl[\int_0^T(\overline{\psi}_t(h))'dU_t\biggr]^2\\
&=\mathcal{J}(\sigma^0,\psi)+\delta\cdot 2\widetilde{z}_0^{-1}E^{\widetilde{Q}}\biggl[\int_0^T(\psi_t(\sigma^0)-\psi_t^H)'dU_t\int_0^T(\overline{\psi}_t(h))'dU_t\biggr]\\
&\qquad+\delta^2\widetilde{z}_0^{-1}E^{\widetilde{Q}}\biggl[\int_0^T(\overline{\psi}_t(h))'dU_t\biggr]^2.
\end{aligned} \tag{3.12}$$

Using Proposition 8 of Rheinländer and Schweizer [4], we have for each h

$$\frac{\widetilde{z}_0}{\widetilde{z}_T}G_T(h,\Theta)=\biggl\{\int_0^T(\overline{\psi}_t(h))'dU_t:\ \overline{\psi}(h)\in L^2(U,\widetilde{Q})\biggr\},$$

and hence, by (3.2),

$$
\begin{aligned}
&E^{\widetilde{Q}}\bigg(\int_0^T (\psi_t(h))' dU_t\bigg)^2 \\
&= E^{\widetilde{Q}} \frac{\widetilde{z}_0^2}{\widetilde{z}_T^2} G_T^2(h,\theta) = \widetilde{z}_0 E G_T^2(h,\theta) = \widetilde{z}_0 E\bigg(\int_0^T \theta_t dR_t^h\bigg)^2 \\
&= \widetilde{z}_0 E\bigg(\int_0^T \theta_t' h_t dM_t^0\bigg)^2 = \widetilde{z}_0 E\bigg(\int_0^T \theta_t' h_t d\langle M\rangle_t k_t + \int_0^T \theta_t' h_t dM_t\bigg)^2 \\
&\le \text{const.}\bigg[E\bigg(\int_0^T |\theta_t' h_t d\langle M\rangle_t k_t|\bigg)^2 + E\bigg(\int_0^T \theta_t' h_t dM_t\bigg)^2\bigg] \\
&\le \text{const.}\, r^2 E\int_0^T \theta_t' d\langle M\rangle_t \theta_t < \infty, \qquad (3.13)
\end{aligned}
$$

as it follows from Remark 2.

Further,

$$
\begin{aligned}
&\bigg(E^{\widetilde{Q}}\bigg[\int_0^T (\psi_t(\sigma^0) - \psi_t^H)' dU_t \int_0^T (\overline{\psi}_t(h))' dU_t\bigg]\bigg)^2 \\
&\le E^{\widetilde{Q}}\bigg(\int_0^T (\psi_t(\sigma^0) - \psi_t^H)' dU_t\bigg)^2 E^{\widetilde{Q}}\int_0^T (\overline{\psi}_t(h))' dU_t\bigg)^2 < \infty, \qquad (3.14)
\end{aligned}
$$

From these estimates we conclude that:

$$
1)\ \ D\mathcal{J}(\sigma^0, h, \psi, \overline{\psi}) = 2\widetilde{z}_0^{-1} E^{\widetilde{Q}} \int_0^T (\psi_t(\sigma^0) - \psi_t^H)' d\langle U\rangle_t \overline{\psi}_t(h) < \infty, \qquad (3.15)
$$

thanks to (3.12), with evident notations in argument of functional $D\mathcal{J}$.

2) $D\mathcal{J}(\sigma^0, h, \psi, \overline{\psi})|_{h\equiv 0} = 0$, since $\overline{\psi}(0) = 0$ by (3.11) and (3.9). Thus

$$
\sup_{h\in\mathcal{H}} D\mathcal{J}(\sigma^0, h, \psi, \overline{\psi}) \ge 0. \qquad (3.16)
$$

3) From (3.14) and (3.13) we get

$$
\begin{aligned}
&\big(D\mathcal{J}(\sigma^0, h, \psi, \overline{\psi})\big)^2 \le \text{const.}\, \widetilde{z}_0^{-2} r^2 \\
&\times E^{\widetilde{Q}} \int_0^T (\psi_t(\sigma^0) - \psi_t^H)' d\langle U\rangle_t (\psi_t(\sigma^0) - \psi_t^H) E\int_0^T \theta_t' d\langle M\rangle_t \theta_t < \infty.
\end{aligned}
$$

Thus $D\mathcal{J}(\sigma^0, h, \psi, \overline{\psi})$ is estimated by the expression which does not depend on h, and is equal to zero if we substitute $\psi_t(\sigma^0) \equiv \psi_t^H$, $0 \le t \le T$.

Hence, by (3.16),

$$0 \le \sup_{h\in\mathcal{H}} D\mathcal{J}(\sigma^0, h, \psi, \overline{\psi})\big|_{\psi\equiv\psi^H} \le \sup_{h\in\mathcal{H}} |D\mathcal{J}(\sigma^0, h, \psi, \overline{\psi})|\big|_{\psi\equiv\psi^H} = 0. \quad (3.17)$$

Further, from (3.16) it follows that we can take $c \in [0, \infty)$ in (3.6).

Now substituting $\psi = \psi^H$ into $\mathcal{J}(\sigma^0, \psi)$ and $D\mathcal{J}(\sigma^0, h, \psi, \overline{\psi})$, we get

$$\mathcal{J}(\sigma^0, \psi^H) = \min_{\psi} \mathcal{J}(\sigma^0, \psi) = \widetilde{z}_0^{-1}(E^{\widetilde{P}}H - x)^2 + \widetilde{z}_0^{-1}E^{\widetilde{Q}}L_T^2$$

(see Lemma 5.1 of Gourieroux et al. [2]) and

$$\sup_{h\in\mathcal{H}} \frac{D\mathcal{J}(\sigma^0, h, \psi^H, \overline{\psi})}{\mathcal{J}(\sigma^0, \psi^H)} = 0.$$

Hence the constraint of problem (3.6) is satisfied.

Remark 4. If $x = E^{\widetilde{P}}H$ and $L_T = 0$, then we get

$$\sup_{h\in\mathcal{H}} \frac{D\mathcal{J}(\sigma^0, h, \psi^H, \overline{\psi})}{\mathcal{J}(\sigma^0, \psi^H)} = \frac{0}{0}\,,$$

which is assumed to be zero, since if we consider the shifted risk functional $\widetilde{\mathcal{J}} = \mathcal{J} + 1$, the optimization problem and the optimal trading strategy will not change, but $D\widetilde{\mathcal{J}}(\sigma^0, h, \psi^H, \overline{\psi}) = D\mathcal{J}(\sigma^0, h, \psi^H, \overline{\psi}) = 0$ and $\widetilde{\mathcal{J}}(\sigma^0, \psi^H) = 1$.

Finally, using Proposition 8 of Rheinländer and Schweizer [4], we arrive at the following

Theorem. *In model* (1.10)*, under conditions* (c.1) *and* (c.2)*, the optimal mean-variance robust trading strategy* (*in the sense of Definition* 1) *is given by the formula*

$$\theta_t^* = ((\sigma_t^0)')^{-1}[\psi_t^{1,H} + \zeta_t(V_t^* - (\psi_t^H)'U_t)], \quad 0 \le t \le T, \quad (3.18)$$

where

$$\psi_t^H = (\psi_t^{0,H}, \psi_t^{1,H}), \quad U_t = \left(\frac{\widetilde{z}_0}{\widetilde{z}_t}, \frac{M_t^0}{\widetilde{z}_t}\,\widetilde{z}_0\right)',$$

$$V_t^* = \frac{\widetilde{z}_0}{\widetilde{z}_t}\left(x + \int_0^t (\psi_t^H)'dU_t\right),$$

ψ_t^H and ζ_t are given by the relations (3.10) *and $\widetilde{z}_T = \widetilde{z}_0 + \int_0^T \zeta_t' dM_t^0$, respectively.*

Such and related problems are considered in the papers [8, 9, 10].

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